250 LECTURES ON MATHEMATICS · PUBLISHED SERIALLY · THREE TIMES EACH MONTH

ISSUE No. 9

PRACTICAL MATHEMATICS

THEORY AND PRACTICE WITH MILITARY AND INDUSTRIAL APPLICATIONS

DIFFERENTIAL EQUATIONS

Linear Equations
Second-Order Equations
Partial Differential Equations

MENSURATION

Fundamental Dimensions
Derived Quantities

Tolerances
Significant Figures
Precision

Measurement of Lengths and Angles
Measuring Instruments of
Areas and Volumes

— ALSO —

Mathematical Tables and Formulas
Self-Tests and Mathematics Problems

KARL MENGER, Ph.D.
University of Notre Dame

35¢

EDITOR: REGINALD STEVENS KIMBALL ED.D.

ISSUE 9 **Practical Mathematics** VOLUME 2

REGINALD STEVENS KIMBALL, Editor

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CHATS WITH THE EDITOR

PRACTICAL MATHEMATICS brings you, in Issue Number Nine, two subjects which round out the theoretical treatment of mathematical principles. In the first of these, differential equations, Dr. Menger assists you to build upon the work in the calculus, presented in Issue Number Eight by Dr. Wiener and Mr. Harvey. In the second, Mr. Baker extends your concept of problems involving measurement and describes for you various instruments with which the practical man is likely to come in contact.

You will probably find that the article on differential equations will demand more in time and concentrated energy than most of the articles which have preceded. When you realize the advantage of being able to solve a differential equation, you will find it worth all the effort of mastering the theory.

The article on mensuration is of more immediate practical utility. Most of us know little about measurement except the "tables" which we half remember from our grammar-school days. In this article, Mr. Baker discusses most of the common instruments which are called into play in achieving precision in measuring distances, quantities, and volumes.

When you have completed the study of this issue, you will have come to a full stop in your acquirement of an over-view of mathematics. From this point on, the issues of PRACTICAL MATHEMATICS become even more practical, helping you to harness the theory to use for specific purposes. It is this second part of the course for which many of you have been waiting eagerly. In the next five issues, we shall discuss construction engineering, machine-shop practice, heat, chemistry, electricity, navigation, aviation, and map-reading. Probably some one of these fields will interest you more than any of the others; for a complete realization of the part mathematics plays in the war effort, you will need to know something about all of them.

Perhaps a word of caution will not be out of place here. The fact that we are winding up the theoretical treatment of mathematics at this point does not mean that we have "finished" all that there is in mathematics. As we have pointed out to you several times before, it has been our aim in this course to select just those points which have been found to be in most demand in connection with entrance into the military and naval services and in industries which are geared to the war effort. At some later time, when you have more leisure, you will find it desirable to extend your knowledge of each field of mathematics, doing more intensively the branches over which we have made a speedy survey in this course.

In geometry, for example, there are many interesting formal proofs which we have completely ignored. In trigonometry, there are many more identities which could be worked out and memorized with great profit. In the calculus, there are boundless possibilities which we have not even touched upon—much less explored. Even in the field of arithmetic, there are a great many problems which did not come within the scope of this course.

No matter how far you may decide

to go in your study of mathematics, you will find it desirable from time to time to refresh your memory and to practice the techniques involved in the fundamental operations. You have probably found time and again that your failure to get correct the solutions to some of the problems in higher mathematics was due entirely to an error in arithmetic. Because of having built up faulty responses for some of the more fundamental steps, even the greatest of us sometimes go astray. I worked at one time with one of the greatest mathematicians this country has ever known. Invariably, he would write on the blackboard "$2 \times 3 = 5$", and only when his class called out that there was an error at that point would he be able to discover why he couldn't make the solution check. If you have any little habits of "faulty association" which are constantly baffling you, work on them until you have eradicated them, lest they cause you delay and embarrassment at moments when every second counts.

For many formulas and values, it is not necessary that they be committed to memory. If you know where to find them and can turn easily to the tables where they are given, you may save yourself the trouble of trying to carry all of the information around in your head. If you have occasion to use them at frequent intervals, however, you will find it worth while to commit them to memory. For instance, if you know the values of the square roots of the common numbers which are always cropping up—like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, . . .—you will be saved the necessity of consulting a table of square roots every time that they occur in a problem. On the other hand, you will seldom need the value of $\sqrt{139}$; it would be foolish to commit it to memory; it is much simpler to use the table in the event that this value crops up in your work. For the work which you are doing, you and you alone are in a position to determine whether or not it is to your advantage to learn a particular formula or value. We have pointed out in the text some of the more common formulas, which are always appearing in problems. For the others, your own experience must be your guide.

Now that we have run through the common branches of mathematics, we shall find it profitable, before beginning the work in the applied fields, to look back over what we have been doing.

In Issue Number One, we gave attention to the fundamental operations in arithmetic, then going on to work with common fractions, a discussion of the number system commonly employed, and a treatment of simple weights and measures.

In Issue Number Two, we carried our treatment of arithmetic through decimals and aliquots, and did some work with denominate numbers (changing weights and measures from one unit of value to another). In this issue, we also did some work with percentages, averages, and ratios. Issue Number Two also included an elementary treatment of logarithms and the slide rule, devices for saving much work in computation.

Issue Number Three gave us an introduction to algebra and a start on the solution of algebraic fractions and equations. We found out how to make use of known facts to arrive at the values of unknowns.

Issue Number Four carried us further into algebra, with techniques for finding the values of several unknowns or for dealing with higher powers of the unknown. In this issue, we learned also how to harness our knowledge of factoring to the problem of finding roots and powers. We considered also the advantages of being familiar with the binomial theorem and of knowing how to deal with num-

bers in series. Finally, we learned how to plot graphs of common equations and how to use graphs for showing the relationships existing among various values.

In Issue Number Five, we considered the axioms and theorems of plane geometry—particularly those dealing with parallel lines, angles, and circles—and learned how to make use of them in proving congruence and finding areas of plane figures. A special article on construction gave attention to the common figures which we frequently find it desirable to know how to draw. The article on the conic sections in this issue acquainted us with the peculiar properties of various curves, such as the ellipse, the hyperbola, and the parabola.

Issue Six was devoted primarily to solid geometry. Here we extended our concept of figures from two-dimensional to three-dimensional objects, and became acquainted with methods of finding volumes and areas of prisms, cylinders, pyramids, cones, and spheres. In this issue, we had also some work in higher algebra, with attention to complex numbers.

Issue Seven introduced us to trigonometry. In addition to a consideration of the utility of trigonometry in solving both right and oblique triangles, we learned how to use the slide rule and the logarithmic tables to cut down the computational work.

In Issue Eight, we learned some elementary calculus and discussed a bit more algebra on which some of the work in the calculus is based.

In the present issue, as we said earlier in this chat, we round out our study of theoretical mathematics with attention to differential equations and mensuration.

With the next issue, we shall be ready to make direct application of these theoretical treatments to specific fields of endeavor. Issue Number Ten will present the mathematics of construction engineering and of machine-shop practice. Mr. Sollenberger and Mr. Benedict are both admirably fitted to guide you into these areas, because they are intimately connected with the war training program in these respects.

If you have been tempted to "slight" any of these matters in your haste to keep up with the course, let us advise you to pause long enough to make sure that you have complete mastery. You will find that your progress through the remaining issues of the course will be the faster because of additional time spent in perfecting your understanding.

Those of you who have been using this course as a review of work previously learned may have been surprised at times at the extent to which we have taken for granted steps which you remember once having been called upon to "prove". Once upon a time, mathematics consisted almost entirely of formal proofs. Just as it is possible to drive an automobile without being able to build one, however, just so it is possible to be able to make use of mathematical principles without being able to derive or justify them. For the man or woman with plenty of time and sufficient interest and inclination, the formal proofs would have proved profitable and entertaining, but not essential to a mastery of techniques. In line with the recommendations of leading educators concerned with getting as many people as possible ready to utilize their mathematical knowledge as quickly as possible, we have omitted the formal aspects of the various branches and confined ourselves to the practical implications.

R.S.K.

ABOUT OUR AUTHORS

OF THE roster of academicians contributing to PRACTICAL MATHEMATICS, Karl Menger is the sole representative of the old world. Born in 1902 in Vienna, Austria, he received his early and advanced education in that city, earning the degree of Doctor of Philosophy at the University of Vienna in 1924. In 1925, he accepted an appointment as lecturer at the University of Amsterdam, Holland, where he remained until 1927. He then decided to return to the University of Vienna, where he served as professor for a ten-year period which was interrupted by two visits to America in 1930 and 1931. On both these occasions, he was a visiting lecturer, first at Harvard and then at Rice Institute.

In 1937, Dr. Menger permanently came to America as Professor of Mathematics at Notre Dame. He has made occasional departures from South Bend on specific assignments, such as during the summer of 1938, when he was visiting professor at the University of California. Menger's extra-curricular activities have been many and varied. He was Vice-President of the International Congress of Mathematics at Oslo in 1936. In 1928, he published *Dimensionstheorie*, and in 1932, *Kurventheorie*. He is the editor of *Reports of a Mathematical Colloquium*, the first series of which he worked on at Vienna from 1928 to 1937, and the second series at Notre Dame since 1938.

Dr. Menger has been a prolific contributor to mathematical journals in this country and abroad. The titles of some of his better known treatises have been "Theory of Dimension and of Curves", "Metric Geometry and Applications of Various Branches of Mathematics", "Algebra of Geometry","Foundations of Mathematics", "Mathematical Theory of Human Relations".

ROBERT H. BAKER, who is the author of the article on measurement in this issue, has been able to project his manuscript in a perspective which will prove particularly interesting to readers of PRACTICAL MATHEMATICS. His experience as Assistant Director of the War Industries Training School of Stevens Institute has provided him with a knowledge of the type of instruction in which this publication is specializing.

The Stevens War Industries Training School is one of the more ambitious and comprehensive attempts of its kind. Sponsored by the Federal Government, it was created in 1942 for the purpose of fitting people for war jobs. It gives a full-time course in introduction to engineering and such specialized courses as metallography, chemistry, powder metallurgy, radio and communications engineering, and the training of inspectors of Army Air Force equipment.

Mr. Baker was born in 1912 in Dedham, Massachusetts. At Harvard, he specialized in the engineering sciences and graduated with a degree of Bachelor of Science in 1935. For a time, he was engaged in the commercial field in the engineering phase of the air-conditioning industry, but in 1938 came to Stevens. Here he served as test administrator and research associate of the Human Engineering Laboratory until 1942. When the War Industries Training School was organized in February of that year, Mr. Baker became its Assistant Director.

· DIFFERENTIAL EQUATIONS ·

By Karl Menger, Ph.D.

TO GAIN an understanding of the meaning of differential equations, let us begin with two simple experiments:

First, cover a thin bar magnet, NS, in a vertical position with a horizontal piece of cardboard, and sprinkle fine iron splinters onto the cardboard. Each splinter, after coming to rest, points in a definite direction (*viz.*, toward the pole, N, of the magnet) and the splinters fit together along lines, called the *lines of force* of the magnet in the plane of the cardboard (*viz.*, the straight lines through N, as in Fig. 1). Next, cover the bar magnet in a horizontal position with a horizontal piece of cardboard and repeat the experiment (Fig. 2). Each splinter will again assume a definite direction depending upon the place at which it comes to rest, and the splinters will fit together along the lines of force of the horizontal magnet in the plane of the cardboard. (In this case, the lines are curved.) These facts are physical illustrations of simple differential equations and their solutions.

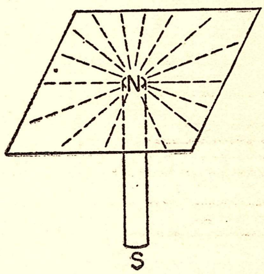

Fig. 1

MEANING OF DIFFERENTIAL EQUATIONS

Geometrically, each simple differential equation describes a *direction field*—that is to say, it associates a direction with each point of the plane or a part of the plane. The solutions of such a differential equation correspond with the curved or straight *lines fitting into the direction field*—that is to say, those lines which at each of their points have the direction which is associated with that particular point. There are many lines fitting into a direction field—one through each point of the direction field.

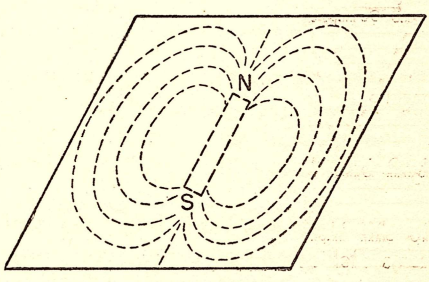

Fig. 2

To obtain an arithmetical description of the situation, we choose a coördinate system in the plane—that is to say, we draw two perpendicular straight lines, OX and OY, and choose a linear unit (say, an inch).

Let P be any given point. If x is the distance (in inches) from P to the line, OY, and y the distance (in inches) from P to the line, OX, then P is called the point with the coördinates (x, y) or, briefly, the point, (x, y).

In order to describe a direction field, we call $s\ (x, y)$ the *slope* of the direction associated with the point (x, y). (Slope of a direction is the tangent of the angle between the direction and the line, OX.)

Now let $y=y(x)$ be the equation of a line and consider a point of this line—say, the one with the coördinates $[x, y(x)]$. The slope of the line at this point is the derivative or the differential quotient of the function, $y(x)$, which is denoted by $y'(x)$ or by $\frac{dy(x)}{dx}$. The slope associated with this point in the direction field is $s[x, y(x)]$.

If the two are to be equal, the function, $y(x)$, has to satisfy the condition,

$$y'(x)=s[x, y(x)] \quad \text{or} \quad \frac{dy(x)}{dx}=s[x, y(x)]. \qquad \mathrm{I}$$

If $y=y(x)$ is a line fitting into the direction field, then the function, $y(x)$, must satisfy condition I for *each* x, and *vice versa*.

The condition, which is frequently written in the abbreviated form,

$$y'=s(x, y) \quad \text{or} \quad \frac{dy}{dx}=s(x, y),$$

is called a *differential equation*—more specifically, a differential equation of the first order for $y(x)$, since only the first derivative of $y(x)$ is involved. Each function, $y(x)$, satisfying equation I is called a *solution* of the differential equation.

Uses of differential equations

At this point, let us consider some applications of differential equations to specific situations.

VERTICAL BAR MAGNET

In the example of the vertical bar magnet, we choose as the point, O, of our coördinate system the pole, N (Fig. 3).

In this case, the direction associated with each point, P, is toward O. Hence, if P has the coördinates (x, y), where $x \neq 0$, then the associated direction has the slope, $s(x, y)=\frac{y}{x}$. If $x=0$—that is to say, if P lies on the axis, OY—then no finite slope is associated with P. Thus, the differential equation reads

$$y'(x)=\frac{y(x)}{x} \quad \text{or, briefly, } y'=\frac{y}{x} \text{ for } x \neq 0. \qquad \mathrm{II}$$

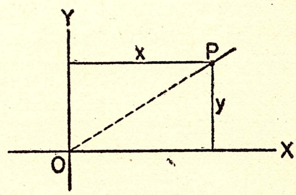

Fig. 3

Let us consider the line, $y=3x$, and any point on it. The slope of the line at that point is 3. The slope, $s(x, y)=\frac{y}{x}$, associated with a point of the line, $y=3x$, is $\frac{3x}{x}=3$. Since the two are equal, the function, $y=3x$, is a solution of

equation II, and, obviously, this is true for the function, $y=kx$, where k is any constant.

Conversely, let $y(x)$ be any function satisfying condition II. Since the derivative, $y'(x)$, of our function is equal to $\frac{y(x)}{x}$, its second derivative, y'', is equal to the derivative of the quotient, $\frac{y(x)}{x}$ —that is to say,

$$y''(x)=\frac{xy'(x)-y(x)}{x^2}.$$

Substituting for $y'(x)$ its expression, $\frac{y(x)}{x}$, we see that

$$y''(x)=\frac{y(x)-y(x)}{x^2}=0$$

for each x.

A function whose derivative is equal to 0 for each x is constant. Since $y'(x)$ is a function with the derivative, $y''(x)$, it follows that $y'(x)$ is constant. We thus have proved that, if $y(x)$ is a solution of II, then for some number, k, we have $y'(x)=k$.

Hence,
$$y(x)=\int k\,dx=kx+c,$$
for some constant, c.

Since $y(x)$ is a solution of II, we have

$$y'(x)=\frac{kx+c}{x}=k+\frac{c}{x}.$$

On the other hand, for the function, $y(x)=kx+c$, we have $y'(x)=k$.

It follows that $c=0$. We thus have proved that, if $y(x)$ is a solution of II, then $y(x)=kx$ for some constant, k.

Altogether, we have found that each function, $y(x)=kx$, is a solution of II, and each solution of II is a function, $y(x)=kx$.

Geometrically, this means that the straight lines through O are the lines fitting into the direction field corresponding to equation II, except the line, OY, which is not the graph of any function, $y=y(x)$. This checks the result of our first experiment, for the lines of force of the vertical bar magnet on the horizontal cardboard are just the straight lines through the pole.

DISINTEGRATION OF RADIUM

A quantity of m grams of radium disintegrates with an instantaneous rate of

$$0.000433m=4.33\cdot10^{-4}m \text{ grams per year —}$$

that is to say, if at any moment we observe m grams, then one year later by means of an accurate scale we shall detect only approximately:

$$m-4.33\cdot10^{-4}m \text{ grams.}$$

The quantities left after half a year or after one month will be approximately

$$m-4.33\cdot10^{-4}\cdot\frac{1}{2}m \text{ grams and } m-4.33\cdot10^{-4}\cdot\frac{1}{12}m \text{ grams,}$$

respectively. Call m_0 the quantity observed today, $m(t)$ the quantity left over after t years. At that time, the instantaneous rate of change of the quantity is $m'(t)$ or $\frac{dm(t)}{dt}$. The law of observation can thus be expressed by the differential equation,

$$m'(t) = -4.33 \cdot 10^{-4} m(t) \text{ for each } t \qquad \text{III}$$

or, in the abbreviated notation,

$$\frac{dm}{dt} = -4.33 \cdot 10^{-4} m.$$

What can we say about the solutions of this equation? Let $m(t)$ be a function satisfying condition III. Then, if $m(t) \neq 0$, we have also

$$\frac{m'(t)}{m(t)} = -4.33 \cdot 10^{-4} \text{ for each } t$$

and hence

$$\int \frac{m'(t)}{m(t)} dt = \int -4.33 \cdot 10^{-4} dt.$$

It follows that

$$\log m(t) = -0.000433t + c,$$

for the derivative of the function, $\log m(t)$, is $\frac{m'(t)}{m(t)}$ and that of $-0.000433t+c$ is $-4.33 \cdot 10^{-4}$. By virtue of the last equality, we can say that the solution, $m(t)$, of equation III must be

$$m(t) = e^{-0.000433t+c} = e^{-0.000433t} \cdot e^c$$

for some constant, c, or

$$m(t) = Ce^{-0.000433t}$$

if we set $C = e^c$.

Conversely, for each constant, C, the function, $Ce^{-0.000433t}$, is a solution of equation III, since the derivative of this function is equal to $-0.000433Ce^{-0.000433t}$, that is, $4.33 \cdot 10^{-4}$ times the function itself. For instance, $3e^{-0.000433t}$ is a solution, $-5e^{-0.000433t}$ is another one.

Now, in addition to the law about the disintegration of radium expressed in formula III, we assume that, at the present moment (*i.e.*, after 0 years), a quantity of m_0 grams is observed. This assumption determines which of the infinitely many solutions of equation III fits our problem: The particular function, $m(t)$, we are looking for has the value, m_0, for $t=0$. Since $e^0=1$, the two solutions mentioned above have the values, 3 and -5, for $t=0$. The solution for which $m(0)=m_0$, is obviously the function,

$$m(t) = m_0 e^{-0.000433t}.$$

This is a function describing how much of the m_0 grams of radium which we observe at present will be left after t years; *e.g.*, we can say: After 5 years, $m_0 e^{-0.0002165}$ grams will be left; after 100 years, $m_0 e^{-0.0433}$ grams.

Our function tells us not only how much radium will be left at a given time but also at which moment a given quantity (say, a grams) will be left. It is the moment at which

$$a = m_0 e^{-0.000433t}; \text{ hence, } \frac{a}{m_0} = e^{-0.000433t},$$

and thus

$$-0.000433t = \log \frac{a}{m_0} = \log a - \log m_0.$$

Consequently,

$$t = \frac{\log m_0 - \log a}{0.000433}.$$

(Here and in the rest of this paper log denotes the natural logarithm to the base, $e = 2.71828183$.)

Of particular interest is the question as to when $\frac{1}{2}m_0$ gram will be left. According to our last formula, this happens at

$$t = \frac{\log m_0 - \log \frac{1}{2}m_0}{0.000433} = \frac{\log m_0 - (\log m_0 - \log 2)}{0.000433}$$

$$= \frac{\log 2}{0.000433} = \frac{0.693}{0.000433} = 1600.$$

We have thus proved that, after 1600 years, exactly $\frac{m_0}{2}$ grams—that is, one-half of the m_0 initial grams—will be left, while one-half will have disintegrated. This period is called the *half life* of radium. Since radium was discovered around 1900, we have not had an opportunity to set up records of the disintegration of radium covering a period of more than 50 years, yet the theory of differential equation III makes us as certain of the fact that in 1600 years exactly one-half of each quantity of radium disintegrates as though we had records of observations.

TEST YOUR KNOWLEDGE OF DISINTEGRATION WITH THESE EXERCISES

1 A quantity of m grams of Polonium disintegrates with an instantaneous rate of $5 \cdot 10^{-3}m$ grams per day. The reader should compute how much of an initial quantity of m_0 grams of Polonium will be left after t days; in particular, after 5 days, after 100 days, after one year. What is the half life of Polonium? After how many days is only $\frac{1}{4}$ of the initial amount left?

2 Suppose that, for a known constant positive number, k, a substance disintegrates according to the law,

$$m'(t) = -k \cdot m(t), \text{ or } \frac{dm}{dt} = -km \qquad \text{IV}$$

Show that, if m_0 grams are present at $t=0$, then $m(t) = m_0 e^{-kt}$ and find the half life.

3 Mesothorium disintegrates according to law IV and has a half life of 6.7 years. Find k and compute how much of 2 grams will be left over after t years.

Suppose now that a substance disintegrates according to law IV—*i.e.*, with a rate of change, $m'(t)$, which is proportional to the quantity, $m(t)$, but that we do not know the numerical value of the factor of proportionality, k.

If at present we find m_0 grams, then it follows (as in problem 2) that

$$m(t)=m_0 \cdot e^{-kt},$$

but k is now not known. However, if we weigh the substance again at some later moment (say, at the moment, T) and find that, at that time, M grams are left, then we have $m(T)=M$ and hence

$$M=m_0 \cdot e^{-kT}.$$

It follows that

$$\frac{M}{m_0}=e^{-kT}.$$

If we draw the Tth root of both expressions of this equality, we obtain

$$e^{-k}=\left(\frac{M}{m_0}\right)^{\frac{1}{T}}.$$

It follows that

$$e^{-kt}=\left(\frac{M}{m_0}\right)^{\frac{t}{T}}$$

and hence

$$m(t)=m_0 \cdot e^{-kt}=m_0\left(\frac{M}{m_0}\right)^{\frac{t}{T}}.$$

For instance, if at present we observe 5 grams of a substance which disintegrates according to law IV, and find 6 weeks later only 4 grams, then after t weeks we shall observe

$$m(t)=5\left(\frac{4}{5}\right)^{\frac{t}{6}} \text{ grams.}$$

In particular, after 12 weeks,

$$5\left(\frac{4}{5}\right)^2=3.2 \text{ grams;}$$

after 24 weeks,

$$5\left(\frac{4}{5}\right)^4=\frac{256}{125}=2.0048 \text{ grams.}$$

The half life will be obtained from the equality,

$$\frac{5}{2}=5\left(\frac{4}{5}\right)^{\frac{t}{6}};$$

thus,

$$t=6\,\frac{\log\frac{1}{2}}{\log\frac{4}{5}}.$$

The factor of proportionality in this case is $k=\frac{1}{6}\log\frac{5}{4}$.

For some values of k $\left(\text{say, for } k=-\frac{1}{2} \text{ and } k=\frac{1}{2}\right)$, the reader should plot the direction field and some lines fitting into the direction field of equation IV.

For this purpose, we should call the solutions $y(x)$ and write the equation,

$$y'(x) = -\frac{1}{2}y(x) \text{ or } \frac{dy}{dx} = -\frac{1}{2}y,$$

and

$$y'(x) = \frac{1}{2}y(x) \text{ or } \frac{dy}{dx} = \frac{1}{2}y.$$

A POPULATION PROBLEM

As another example, let us consider a population with the birth rate of b per 1000 per year and a death rate of a per 1000 per year.

At any moment, the instantaneous rate of change of the population is $\frac{b-a}{1000}$ times the population. If $p(t)$ is the population t years from now, then for each t we have

$$\frac{dp}{dt} = \frac{b-a}{1000}p \text{ or } p'(t) = \frac{b-a}{1000}p(t)$$

and thus

$$\frac{p'(t)}{p(t)} = \frac{b-a}{1000}.$$

It follows that $\int \frac{p'(t)}{p(t)}dt$ and $\int \frac{b-a}{1000}dt$ differ only by a constant—that is to say,

$$\log p(t) = \frac{b-a}{1000}t + c$$

and, consequently,

$$p(t) = e^{\frac{b-a}{1000}t+c} = e^c \cdot e^{\frac{b-a}{1000}t}.$$

If at present the population is p_0, then $p_0 = e^c \cdot e^0 = e^c$. Since $e^c = p_0$, we have

$$p(t) = p_0 \cdot e^{\frac{b-a}{1000}t}.$$

We have to distinguish three cases:

a Birth rate and death rate are equal. Then $b-a=0$ and $p(t)=p_0$ for each t. That is to say, the population is stationary.

b The death rate exceeds the birth rate. Then $b-a$ is negative and the population decreases in a way similar to the disintegration of radium. In particular, we can find out after how many years the population will decrease to half of its present number, a problem corresponding to the determination of the half life of radium. We have to solve for t the equality,

$$\frac{1}{2}p_0 = p_0 e^{\frac{b-a}{1000}t} \text{ or } \frac{1}{2} = e^{\frac{b-a}{1000}t}.$$

We obtain

$$\frac{b-a}{1000}t = \log\frac{1}{2} \text{ and } t = \frac{1000}{b-a}\log\frac{1}{2}.$$

In this expression, $b-a$ and $\log \frac{1}{2}$ are negative; thus, t is positive. If $b-a=-1$ (that is to say, if in a population the number of deaths for each 1000 per year exceeds the number of births by 1), then $t=\frac{1000}{-1}\log\frac{1}{2}=1000\log 2$ (that is, approximately 693) and hence in about 693 years the population will decline to half of its present number. If $b-a=-2$, then this decline will occur in 347 years; if $b-a=-6$, in 115 years.

c The birth rate exceeds the death rate. Then $\frac{b-a}{1000}$ is positive and $p(t)$ will rapidly increase. In order to find out after how many years the population will double, we have to solve the equation,

$$2p_0=p_0e^{\frac{b-a}{1000}t} \text{ or } 2=e^{\frac{b-a}{1000}t} \text{ for } t.$$

We obtain

$$\frac{b-a}{1000}=\log 2 \text{ and } t=\frac{1000}{b-a}\log 2.$$

We see that, if the birth rate per 1000 per year exceeds the death rate by 1, 2, or 6, then the population will double in about 693, 347, or 115 years, respectively. The reader may prove the general statement that, if the birth rate exceeds the death rate by n, it takes the population just as long to double as it takes it to decline to one-half, if the death rate exceeds the birth rate by n.

SOLVING DIFFERENTIAL EQUATIONS

As the next example, let us consider the equation, $y'(x)=s[x,y(x)]$ in the case when $s(x,y)$ is the quotient of a function, $f_1(x)$ of x alone, and a function $f_2(y)$ of y alone.

Then the equation reads as follows:

$$y'(x)=\frac{f_1(x)}{f_2[y(x)]} \qquad \text{V}$$

Familiar laws

If $f_1(x)=\frac{1}{x}$ and $f_2(y)=\frac{1}{y}$, we obtain equation II (*cf.* the vertical bar magnet).

If $f_1(x)=-k$ and $f_2(y)=\frac{1}{y}$, then V becomes equation IV in the form mentioned on page 517. We shall now study further examples of equation V.

First Example:
$$y'(x)=-\frac{x}{y(x)}. \qquad \text{VI}$$

This is the case, $f_1(x)=-x$, $f_2(y)=y$. We obtain

$$\int y(x)y'(x)dx=\int -x dx+c.$$

Now

$$\int -x\,dx=-\frac{x^2}{2} \text{ and } \int y(x)y'(x)dx=y^2(x).$$

Thus,

$$y^2(x) = -x^2 + c^2 \text{ or } x^2 + y^2(x) = c^2.$$

The reader should plot the direction field corresponding to this equation and verify that the circles about the point, (0,0), are the solutions,

Second Example: $$y'(x) = \frac{x}{y(x)} \text{ or, briefly, } \frac{dy}{dx} = \frac{x}{y}. \qquad \text{VII}$$

This is the case, $f_1(x) = x$, $f_2(y) = y$. The solutions are hyperbolae.

Third Example: $$w'(t) = k[w(t) - a] \text{ or, briefly, } \frac{dw}{dt} = k(w - a) \qquad \text{VIII}$$

where k and a are given constants. For a change, we call the function for which we are looking $w(t)$ instead of $y(x)$. In this case, $f_1(t) = k$ and $f_2(w) = w - a$.

Let $w(t)$ be the temperature of a substance at the moment, t. If the substance is in a medium of a lower temperature, a, then, according to Newton, the temperature of the substance decreases with a rate, $w'(t)$, proportional to the difference between $w(t)$ and a. If we call k the factor of proportionality, then VIII expresses Newton's cooling law.

Fourth Example: $$y'(t) = \sqrt{2g\,y(t)}. \qquad \text{IX}$$

If we say $y(t)$ is the number of feet traversed by a falling body during the first t seconds after it was dropped, then $y'(t)$ denotes the velocity of the falling body t seconds after it was dropped. If $g = 32.2$ (thus, $\sqrt{2g}$ approximately equals 8), then IX expresses one of Galileo's laws concerning falling bodies. In order to solve it, we write

$$\frac{y'(t)}{\sqrt{y(t)}} = \sqrt{2g} \text{ and } \int \frac{y'(t)}{\sqrt{y(t)}}dt = \int \sqrt{2g}\,dt + c.$$

Thus, we obtain

$$2\sqrt{y(t)} = \sqrt{2g}\,t + c$$

and hence

$$y(t) = \frac{g}{2}(t-a)^2 \text{ if we set } a = \frac{-c}{\sqrt{2g}}.$$

If we drop the body at $t = 0$, then the initial displacement $y(0) = 0$. Now $y(0) = \frac{g}{2}(0-a)^2$. Since this expression is equal to 0, it follows that $a = 0$ and thus $y(t) = \frac{g}{2}t^2$.

HORIZONTAL BAR MAGNET

We return to the field of a thin horizontal bar magnet, NS, in a horizontal plane as shown in Fig. 2.

In this horizontal plane, we choose a coördinate system in the following way: as the point, O, we choose the center of the bar, NS; as the line, OY, the straight line passing through the poles of the magnet; as OX, the line perpendicular to OY (Fig. 4). Besides, we choose a length unit.

Under the influence of the magnet, everywhere in the plane, an iron splinter, $N'S'$, assumes a definite direction—that is to say, the line, $N'S'$, forms a definite angle with OX. This angle depends, of course, upon the location of the splinter. If the center of the splinter, $N'S'$, is at the point, P, with the

coördinates, (x, y), then we call Q the point with the coördinates $\left(\frac{x}{2}, y\right)$ and β the angle, QOY. From the laws of magnetic attraction, it follows that the splinter, $N'S'$, forms the angle, β, with the line, OP.

Consequently, if we call α the angle, XOP, then the angle between the line, $S'N'$, and OX is $\alpha-\beta$. The slope of the line, $S'N'$, is $\tan(\alpha-\beta)$. Now, from

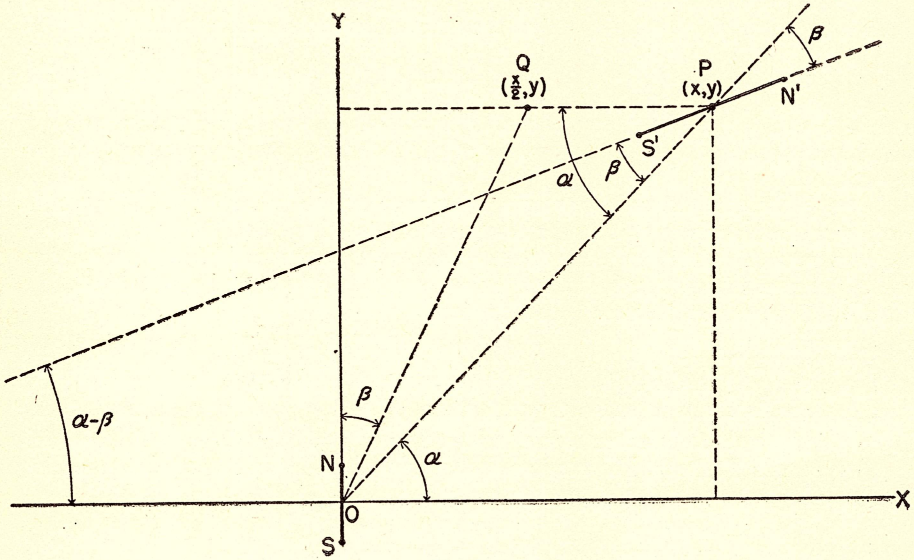

Fig. 4

trigonometry, we know that $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$. Moreover, we see that $\tan\alpha=\frac{y}{x}$ and $\tan\beta=\frac{x}{2y}$. Thus,

$$\tan(\alpha-\beta)=\frac{\frac{y}{x}-\frac{x}{2y}}{1+\frac{1}{2}}=\frac{2}{3}\left(\frac{y}{x}-\frac{1}{2}\frac{x}{y}\right).$$

Denoting by $s(x,y)$ the slope of the splinter whose center is at (x,y), we thus have $s(x,y)=\frac{2}{3}\frac{y}{x}-\frac{1}{3}\frac{x}{y}$. A line of force of the magnet is a line, $y=y(x)$, which for each x at the point, $[x,y(x)]$, has the slope, $s[x,y(x)]$. In other words, a line of force is a curve, $y=y(x)$, such that for each x we have $y'(x)=s[x,y(x)]$—that is,

$$y'(x)=\frac{2}{3}\frac{y}{x}-\frac{1}{3}\frac{x}{y}. \qquad \text{X}$$

Instead of looking for the solutions, $y(x)$, of this differential equation, we shall try to find $\frac{y(x)}{x}$. We shall denote this auxiliary function by $z(x)$. Since, by definition, $z(x)=\frac{y(x)}{x}$, we have $y(x)=x\cdot z(x)$ and consequently, $y'(x)=x\cdot z'(x)+$

$z(x)$. Now $y'(x)$ is the expression on the left side of equation X. The expression on the right side of X can be written as $\frac{2}{3}z(x)-\frac{1}{3z(x)}$. We see, therefore, that, if $y(x)$ satisfies condition X, then the auxiliary function, $z(x)$, satisfies the equation,

$$x \cdot z'(x)+z(x)=\frac{2}{3}z(x)-\frac{1}{3z(x)}$$

and hence the equation,

$$z'(x)=-\frac{1}{3x}\left[z(x)+\frac{1}{z(x)}\right]. \qquad \textbf{XI}$$

This differential equation, XI, is of form V except that the unknown function is denoted by $z(x)$ instead of by $y(x)$. In fact, another way of writing XI is

$$\frac{z(x)\,z'(x)}{z^2(x)+1}=-\frac{1}{3x}.$$

Multiplying both sides of this equation by 2, we see that the two functions,

$$\frac{2z(x)\,z'(x)}{z^2(x)+1} \text{ and } -\frac{2}{3x} \text{ are equal.}$$

It follows that two functions with these two functions as derivatives differ merely by a constant. Now

$$\int\frac{2z(x)z'(x)}{z^2(x)+1}dx=\log\,[z^2(x)+1] \text{ and } \int-\frac{2}{3x}dx=-\frac{2}{3}\log x.$$

Consequently, we can say that for some constant, c,

$$\log\,[z^2(x)+1]=\log x^{-\frac{2}{3}}+c=\log C\,x^{-\frac{2}{3}}$$

if we set $C=e^c$. The constant, C, is therefore positive. From the last equality, it follows that

$$z^2(x)+1=C\,x^{-\frac{2}{3}}.$$

Since $z(x)=\frac{y(x)}{x}$, we obtain

$$y^2(x)+x^2=C\,x^{\frac{4}{3}}$$

and $\qquad$ **XIIa**

$$y^2(x)=C\,x^{\frac{4}{3}}-x^2.$$

Hence

$$y(x)=+\sqrt{Cx^{\frac{4}{3}}-x^2} \text{ or } y(x)=-\sqrt{Cx^{\frac{4}{3}}-x^2} \qquad \textbf{XIIb}$$

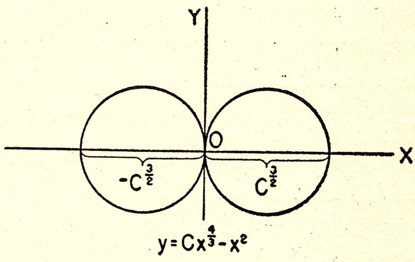

Fig. 5

for some positive constant, C. For each positive C, the curve XII has the shape of the sign, ∞, with the cross point at O (Fig. 5). The two halves of the curve described in XIIb are symmetric with respect to the line, OX. For each positive C, the number, $y(x)$, is defined only if $Cx^{\frac{4}{3}}-x^2\geq 0$. This is the case if $Cx^{\frac{4}{3}}\geq x^2$ or $C\geq x^{\frac{2}{3}}$—that is to say, for $-C^{\frac{3}{2}}\leq x\leq C^{\frac{3}{2}}$.

TEST YOUR ABILITY TO SOLVE DIFFERENTIAL EQUATIONS

The reader can solve the following differential equations by first looking for the auxiliary function, $z(x)=\frac{y(x)}{x}$:

4 $y'(x)=\frac{y^3}{x^3}$ $\qquad$ 5 $y'(x)=e^{\frac{y}{x}}+\frac{y}{x}$ $\qquad$ 6 $y'(x)=\frac{y^2}{x^2}$

ELECTRIC CIRCUITS

We consider an electric circuit containing inductance and resistance.

Let the self-inductance be L henries, the resistance R ohms. We assume that $L \neq 0$. Let $i(t)$ amps be the current t seconds from now. If no electromotive force (e.m.f.) is impressed on the circuit, then the current satisfies the differential equation,

$$L\frac{di}{dt}+Ri=0; \text{ that is, } Li'(t)+Ri(t)=0 \text{ for each } t. \qquad \textbf{XIII}$$

This equation can be written in the form,

$$i'(t)=-\frac{R}{L}\,i(t),$$

which is of the type of equation IV. Its solutions are $Ce^{-\frac{R}{L}t}$ where C is a constant. If at $t=0$ the current is i_0 amps, then

$$i(t)=i_0\,e^{-\frac{R}{L}t}.$$

In particular, if $R=0$, the current will be stationary, $i(t)=i_0$ for each t.

Suppose now that an alternating e.m.f. is impressed on our circuit (say, at the moment t); then the e.m.f.,

$$E(t)=E\cos 2n\pi t \text{ or, briefly, } E(t)=E\cos\alpha t.$$

E is the maximum voltage, $n=\frac{\alpha}{2\pi}$ the number of cycles per second. Then $i(t)$ satisfies the differential equation,

$$L\frac{di}{dt}+Ri=E\cos\alpha t. \qquad \textbf{XIV}$$

We shall try to find a solution which is a cosine function with the same period as the e.m.f. Tentatively, we set

$$i(t)=F\cos(\alpha t+\varphi).$$

Then

$$Li'(t)+Ri(t)=-LF\alpha\sin(\alpha t+\varphi)+RF\cos(\alpha t+\varphi).$$

If $i(t)$ is to be a solution of XIV, we must have

$$-LF\alpha\sin(\alpha t+\varphi)+RF\cos(\alpha t+\varphi)=E\cos\alpha t \text{ for each } t. \qquad \textbf{XV}$$

If XV holds for $t=\frac{\pi}{2\alpha}$, we obtain

$$-LF\alpha\sin\left(\frac{\pi}{2}+\varphi\right)+RF\cos\left(\frac{\pi}{2}+\varphi\right)=0;$$

hence,

$$\tan\left(\frac{\pi}{2}+\varphi\right)=\frac{R}{L\alpha} \text{ and } \tan\varphi=-\frac{L\alpha}{R}$$

if $\alpha\neq 0$ and $R\neq 0$; then

$$\varphi=\operatorname{arc\,tan}\left(-\frac{L\alpha}{R}\right)=-\operatorname{arc\,tan}\frac{L\alpha}{R}.$$

If XV holds for $t=-\frac{\varphi}{\alpha}$, then

$$RF=E\cos(-\varphi)=E\cos\varphi=\frac{E}{\sqrt{1+\tan^2\varphi}}.$$

Substituting for tan φ the expression, $-\frac{L\alpha}{R}$, we obtain

$$F=\frac{E}{\sqrt{R^2+L^2\alpha^2}}.$$

We see: If $\alpha \neq 0$ and a function of the form, $F\cos(\alpha t+\varphi)$, is a solution of XIV, then it must be the function,

$$i(t)=\frac{E}{\sqrt{R^2+L^2\alpha^2}}\cos\left(\alpha t-\arc\tan\frac{L\alpha}{R}\right) \text{ if } R\neq 0 \qquad \textbf{XVIa}$$

and similarly

$$i(t)=-\frac{E}{L\alpha}\cos\left(\alpha t-\frac{\pi}{2}\right)=\frac{E}{L\alpha}\sin\alpha\, t \text{ if } R=0. \qquad \textbf{XVIb}$$

By substitution in XIV, the reader can verify that XVI is indeed a solution. Moreover, it is easily seen that each solution of XIV is of the form,

$$i(t)=\frac{E}{\sqrt{R^2+L^2\alpha^2}}\cos\left(\alpha t-\text{arc}\tan\frac{L\alpha}{R}\right)+Ce^{-\frac{R}{L}t} \text{ if } R\neq 0$$

or
$$i(t)=\frac{E}{L\alpha}\sin\alpha\, t+C \text{ if } R=0 \qquad \textbf{XVII}$$

where C is some constant.

The reader should determine for which constant we obtained $i(0)=i_0$. As a numerical example, he may treat the case, $n=60$; thus $\alpha=120\pi$, $E=50$ volts $L=5$ henries, $R=20$ ohms. As initial value of $i(t)$, one may choose, for instance, $i_0=0$ or $i_0=1$ amp.

Solution XVI was derived for an alternating e.m.f.; that is, for the case $\alpha\neq 0$. Formulæ XVIa and XVIb suggest the following solution if $\alpha=0$; that is, in the case of a constant e.m.f.:

$$i(t)=\frac{E}{R} \text{ if } R\neq 0, \text{ and } i(t)=\frac{E}{L}t \text{ if } R=0. \qquad \textbf{XVIII}$$

The reader should verify that the functions in XVIII are actually solutions of the equations,

$$Li'(t)+Ri(t)=E \text{ (for } R\neq 0) \text{ and } Li'(t)=E, \text{ respectively.} \qquad \textbf{XIX}$$

and that each solution of XIX is

$$i(t)=\frac{E}{R}+Ce^{-\frac{R}{L}t} \text{ and } i(t)=\frac{E}{L}t+C$$

for some constant, C. Moreover, he should determine that solution which for $t=0$ assumes the value, i_0.

LINEAR EQUATIONS

The differential equations for $i(t)$ studied in the preceding section are called linear because only the first power of the unknown function, $i(t)$, and its first derivative, $i'(t)$, are involved. If $f(x)$ and $g(x)$ are any two given functions,

$$y'(x)+f(x)y(x)+g(x)=0 \qquad \textbf{XX}$$

is called a linear equation for $y(x)$. On the other hand, the equation,

$y'(x)+y^2(x)=0$, is not linear since it contains the square of the unknown function.

In order to find the solutions of XX, consider a function, $y(x)$, satisfying XX which is the product of two functions, $u(x)$ and $v(x)$. If $y(x)=u(x)\cdot v(x)$ satisfies XX, then

$$u(x)\cdot v'(x)+u'(x)\cdot v(x)+f(x)\cdot u(x)\cdot v(x)+g(x)=0;$$

thus,

$$u(x)\,[v'(x)+f(x)\cdot v(x)]+u(x)\cdot v'(x)+g(x)=0. \qquad \text{XXI}$$

If we determine $v(x)$ in such a way that

$$v'(x)+f(x)\cdot v(x)=0, \qquad \text{XXII}$$

then the equation becomes

$$u(x)\cdot v'(x)+g(x)=0, \qquad \text{XXIII}$$

and

$$v'(x)=-\frac{g(x)}{u(x)}.$$

Hence

$$v(x)=-\int\frac{g(x)}{u(x)}dx+c.$$

In order to determine $v(x)$ in such a way that XXII holds, we merely have to solve the differential equation,

$$v'(x)=-f(x)\cdot v(x).$$

The latter equation is of type V. We obtain

$$\int\frac{v'(x)}{v(x)}dx=-\int f(x)dx+c,$$

$$\log v(x)=-\int f(x)dx+c,$$

$$v(x)=e^{-\int f(x)dx+c}.$$

As an example, we treat the case,

$$Li'(t)+Ri(t)=Ee^{-kt}\quad (L\neq 0), \qquad \text{XXIV}$$

describing the current in a circuit with self-inductance L, resistance R, and an exponentially decreasing impressed e.m.f. If $i(t)=u(t)\cdot v(t)$, then

$$Lu(t)v'(t)+Lu'(t)v(t)+Ru(t)v(t)=Ee^{-kt};$$

thus

$$u(t)\cdot[L\cdot v'(t)+Rv(t)]+Rv(t)u'(t)=Ee^{-kt}. \qquad \text{XXV}$$

We can determine $v(t)$ in such a way that

$$Lv'(t)+Rv(t)=0,$$

for we know that the solutions of this differential equation are

$$v(t)=Ce^{-\frac{R}{L}t}.$$

Substituting this function into XXV, we obtain

$$RCe^{-\frac{R}{L}t}u'(t)=Ee^{-kt};$$

hence,

$$u'(t)=\frac{E}{RC}e^{\left(\frac{R}{L}-k\right)t}.$$

Consequently,

$$u(t)=\frac{E}{RC\left(\frac{R}{L}-k\right)}e^{\left(\frac{R}{L}-k\right)t}+D.$$

We find $i(t)$ by multiplying $u(t)$ and $v(t)$. If we set $DC=c$, we obtain

$$i(t)=\frac{E}{R\left(\frac{R}{L}-k\right)}e^{-kt}+ce^{-\frac{R}{L}t}.$$

By substituting this function into XXIV, we verify that for each c it is a solution.

TEST YOUR ABILITY TO SOLVE LINEAR DIFFERENTIAL EQUATIONS

7 Treat equation XIV by the method explained in this section.

8 Treat the equation, $y'+ay+bx+c=0$, where a, b, c are any constants.

Finding an approximation to a particular solution

While we might list more examples of equations which can be solved, no methods are known which would enable us to set up an expression for the solutions of the general differential equation, I. Fortunately, in practical applications we are faced with the problem of finding a particular solution of a differential equation rather than with the task of setting up an expression for all its solutions. Moreover, for many practical purposes it is sufficient to find approximate values of a particular solution. We shall show in a very simple case how this can be achieved.

We shall try to find that solution $y(x)$ of the equation,

$$y'(x)=+2\sqrt{y(x)} \qquad \textbf{XXVI}$$

which for $x=2$ assumes the value, 1. More specifically, we wish to find the approximate value which this solution assumes for $x=2.5$.

We know of the function, $y(x)$, that $y(2)=1$. Thus, by XXVI, we have $y'(2)=2\sqrt{1}=2$. From $y'(2)=2$ it follows that

$$\frac{y(2+\Delta x)-y(2)}{\Delta x}\sim 2 \text{ for small } \Delta x \qquad \textbf{XXVII}$$

where $\sim$ denotes "approximately equal". For $x=0.5$, we obtain

$$\frac{y(2.5)-y(2)}{0.5}\sim 2 \text{ and } y(2.5)\sim y(2)+1=2.$$

We get a second better approximation by applying XXVII to $\Delta x=0.25$, thus obtaining

$$y(2.25)\sim y(2)+0.5=1.5.$$

From this equality, it follows by XXVI that

$$y'(2.25)=2\sqrt{1.5}\sim 2.45.$$

Hence,

$$\frac{y(2.25+\Delta x)-y(2.25)}{\Delta x}\sim 2.45 \text{ for small } \Delta x.$$

Substituting $\Delta x=0.25$, we arrive at

$$y(2.5)\sim y(2.25)+0.613=2.113,$$

which comes closer to the truth than $y(2.5)\sim 2$.

A still better third approximation is obtained by applying XXVI to $\Delta x=0.1$, thus obtaining $y(2.1)\sim 1.2$, and hence

$$y'(2.1)=2\sqrt{1.2}=2.191.$$

Hence, $\dfrac{y(2.2)-y(2.1)}{0.1}\sim 2.191$ and thus $y(2.2)\sim 1.419$. Consequently,

$$y'(2.2)\sim 2\sqrt{1.419}\sim 2.380.$$

Hence, $\dfrac{y(2.3)-y(2.2)}{0.1}\sim 2.380$ and thus $y(2.3)\sim 1.657$. Consequently,

$$y'(2.3)\sim 2\sqrt{1.657}\sim 2.574.$$

Hence, $\dfrac{y(2.4)-y(2.3)}{0.1}\sim 2.574$ and thus $y(2.4)\sim 1.914$. Consequently,

$$y'(2.4)\sim 2\sqrt{1.914}\sim 2.767.$$

Hence, $\dfrac{y(2.5)-y(2.4)}{0.1}\sim 2.767$ and thus $y(2.5)\sim 2.191$.

Since equation XXVI is of the type of equation V, we can find an expression for a family of solutions: $y(x)=(x-c)^2$. The solution which for 2 assumes the value, 1, is $y(x)=(x-1)^2$. For $x=2.5$, this solution assumes the value, $y(2.5)=1.5^2=2.25$. The results of our three approximations (2, 2.113, 2.191) came closer and closer to the truth. The reader should verify that a fourth approximation using 10 steps with $\Delta x=0.05$ (instead of the 5 steps with $\Delta x=0.1$ of the third approximation) will lead to a number still closer to the true value, 2.25, than 2.191.

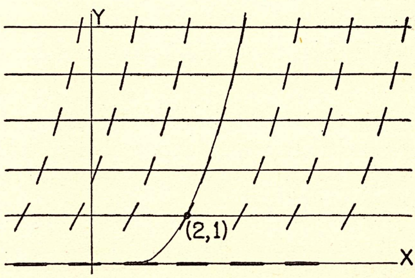

Fig. 6

Furthermore, the reader should study the geometric meaning of our approximations. Fig. 6 shows the direction field of equation XXVI. Consider the point, (2,1). The curve fitting into the field which passes through this point is the parabola, $y=(x-1)^2$. This parabola is replaced by its tangent at the point, (2,1), in the first approximation, and by polygons of 2, 5, and 10 sides in the second, third, and suggested fourth approximations, respectively.

TEST YOUR ABILITY TO APPROXIMATE SOLUTIONS

9 Apply the method to another equation which can be solved otherwise, like $y'(x)=\frac{1}{2}y(x)$, and then to another equation which you cannot solve otherwise.

Several solutions through the same point

We saw that a differential equation, $y'(x)=s[x,y(x)]$, admits many solutions. However, in the cases studied so far we found only one solution, $y(x)$, which for a definite number, x_0, assumes a definite value, y_0. This fact permitted us to determine unambiguously the quantity of radium which will be observed t years from now if we know the present amount, and the current t seconds from now, if we know its present intensity.

The situation is not always so simple as that. The condition,

$$y'^2(x)=4y(x), \qquad \text{XXVIII}$$

admits the solutions, $y(x)=(x-c)^2$ for any constant, c. If we require that $y(2)=1$, then $c=2+1$ or $c=2-1$. In fact, both $y(x)=(x-1)^2$ and $y(x)=(x-3)^2$ satisfy XXVIII and assume the value, 1, for $x=2$. However, XXVIII is really a pair of differential equations, *viz.*, the equation,

$$y'(x)=+2\sqrt{y(x)}$$

studied in the preceding section, and the equation,

$$y'(x)=-2\sqrt{y(x)}.$$

The former has the solution, $y(x)=(x-1)^2$, the latter the solution, $y(x)=(x-3)^2$, assuming the value 1 for $x=2$. Geometrically, condition XXVIII associates two directions with each point, (x, y), for which $y>0$. With the point, (2, 1), it associates the directions with the slopes, $4\cdot\sqrt{1}=4$ and $-4\cdot\sqrt{1}=-4$. Condition XXVIII thus describes two superposed direction fields.

Even if with each point only one direction is associated, several curves fitting into the direction field may pass through the same point. As an example, we consider the differential equation,

$$y'(x)=y^{\frac{2}{3}}(x). \qquad \text{XXIX}$$

One readily verifies that, for each constant, c, the function,

$$y(x)=\frac{(x+c)^3}{27} \qquad \text{XXX}$$

is a solution of XXIX, but the converse is not true. There is a solution of XXIX which for no constant, c, is of form XXX—*viz.*, the function, $y(x)=0$, for each x. In particular, there are two different solutions of XXIX, both assuming the value, 0, for $x=0$—*viz.*, $y(x)=\frac{x^3}{27}$ and $y(x)=0$. Geometrically,

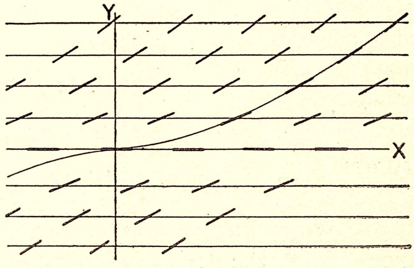

Fig. 7

solutions XXIX are cubic parabolæ, each having the slope, 0, where it crosses the X-axis (Fig. 7). The function, $y(x)=0$, represents the X-axis, which likewise has the slope, 0.

DIFFERENTIAL EQUATIONS OF SECOND ORDER

If we require of a function, $y(x)$, that for each x its second derivative, $\frac{d^2y}{dx^2}$ or $y''(x)$, should be related to x, $y(x)$, $y'(x)$ in a definite way, then we say that $y(x)$ has to satisfy a second order differential equation. The simplest such equation is

$$y''(x)=0. \qquad \textbf{XXXI}$$

We solved it on page 515 by observing that, if $y''(x)=0$, then $y'(x)=k$ for some constant, k, and consequently $y(x)=kx+c$ for some constants, k and c. Conversely, for each two constants, k and c, the function, $y(x)=kx+c$, is a solution of XXXI, since the second derivative of $kx+c$ is 0 for each x.

Suppose now that, for a given number, k, a function is subjected to the condition that for each x the second derivative, $y''(x)$, satisfies the equation,

$$y''(x)=ky(x) \text{ or } \frac{d^2y(x)}{dx^2}=ky(x) \text{ or, briefly, } y''=ky. \qquad \textbf{XXXIIa}$$

Setting $-k=b$, we can write XXXIIa in the form,

$$y''(x)+by(x)=0. \qquad \textbf{XXXIIb}$$

As an example, we study the case, $k=-9$; that is to say, the equation,

$$y''(x)=-9y(x) \text{ or } y''(x)+9y(x)=0. \qquad \textbf{XXXIII}$$

The function, $y(x)=\sin 3x$, is a solution, for if $y=\sin 3x$, then $y'(x)=3\cos 3x$ and $y''(x)=-9\sin 3x$ and hence $y''(x)=-9y(x)$. The reader may verify that $y(x)=5\sin 3x$ and $y=2\cos 3x$ are also solutions, and, more generally, that for each two constants, A and B, the function, $y(x)=A\cos 3x+B\sin 3x$, is a solution of equation XXXIII.

Now let $y(x)$ be any given solution of equation XXXIII. Then we have

$$y''(x)=-9y(x)$$

and hence,

$$2y'(x)y''(x)=-9\cdot 2y(x)y'(x).$$

It follows that a function with the derivative, $2y'(x)y''(x)$, and a function with the derivative, $-9\cdot 2y(x)y'(x)$ differ only by a constant. A function with the derivative, $2y'(x)y''(x)$, is $[y'(x)]^2$, for which we briefly write $y'^2(x)$; a function with the derivative, $-9\cdot 2y(x)y'(x)$, is $-9y^2(x)$. Thus,

$$y'^2(x)=-9y^2(x)+c$$

for some constant, c. Now for each x the number, $y^2(x)$, is certainly non-negative; hence, $-9y^2(x)$ is non-positive. Since $-9y^2(x)+c$ is equal to $y'^2(x)$, which is non-negative, it follows that the constant, c, is non-

negative and thus is the square of a real number (say, of C). In view of $c=C^2$, we can write

$$y'^2(x)=C^2-9y^2(x)$$

and

$$y'(x)=\pm\sqrt{C^2-9y^2(x)}=\pm3\sqrt{\frac{C^2}{9}-y^2(x)}.$$

Hence,

$$\frac{y'(x)}{\sqrt{\frac{C^2}{9}-y^2(x)}}=3.$$

Reasoning as before, we conclude that $\int\frac{y'(x)}{\sqrt{\frac{C^2}{9}-y^2(x)}}dx$ and $\int 3dx$ differ only by a constant. That is to say, arc sin $\frac{3y(x)}{C}$ and $3x$ differ only by a constant, which we shall call D. From

$$\text{arc sin }\frac{3y(x)}{C}=3x+D,$$

it follows that

$$\frac{3y(x)}{C}=\sin(3x+D)$$

and

$$y(x)=\frac{C}{3}\sin(3x+D)=\frac{C}{3}\sin D\cos 3x+\frac{C}{3}\cos D\sin 3x.$$

We have thus proved: If $y(x)$ is any solution of equation XXXIII, then for two constants, A and B, the function, $y(x)$, is equal to $A\cos 3x+B\sin 3x$; namely, for the constants, $A=\frac{C}{3}\sin D$ and $B=\frac{C}{3}\cos D$. The reader had previously verified that for each two constants, A and B, the function,

$$y(x)=A\cos 3x+B\sin 3x,$$

is a solution of equation XXXIII. We have thus obtained a complete survey of the solutions of the second order differential equation, XXXIII.

Exactly in the same way, we get a survey of the solutions of equation XXXII if k is any negative number. In this case, $-k=b$ is positive. For each two constants, A and B, the function, $A\cos\sqrt{b}x+B\sin\sqrt{b}x$ is a solution of the equation, and each solution of the equation is a function, $A\cos\sqrt{b}x+B\sin\sqrt{b}x$, for two constants, A and B. Instead of $\sqrt{b}$, we may write $\sqrt{-k}$.

If k is a positive number, then, by a reasoning similar to the one employed in the case of negative k, one can show that, for each two constants, A and B, the function,

$$y(x)=Ae^{\sqrt{k}x}+Be^{-\sqrt{k}x},$$

is a solution of the equation, and that, conversely, each solution of the equation is of this form. Instead of $\sqrt{k}$ we may write $\sqrt{-b}$.

In the case, $k=0$, equation XXXII is nothing but equation XXXI, whose solutions are $y(x)=Ax+B$.

Summarizing, we can say: The solutions of the equations, XXXIIa and XXXIIb, are

$$y(x)=A\cos\sqrt{b}x+B\sin\sqrt{b}x=A\cos\sqrt{-k}x+B\sin\sqrt{-k}x \quad \text{if } k<0 \text{ and } b>0,$$

$$y(x)=A\,e^{\sqrt{k}x}+B\,e^{-\sqrt{k}x} \qquad =A\,e^{\sqrt{-b}x}+B\,e^{-\sqrt{-b}x} \quad \text{if } k>0 \text{ and } b<0,$$

$$y(x)=A\,x+B \quad \text{if } k=0.$$

Equation XXXII is of great practical importance in mechanics as well as in the theory of electricity.

An example is the movement of a particle fixed to the end of an elastic spring. We choose an axis, OY, through the axis of the spring, and the origin, O, at that place where we find the end of the spring in the neutral position of the latter—that is, when the spring is neither extended nor compressed (Fig. 8). Since the movement of the particle takes place in the straight line, OY, it is not necessary for our present purpose to introduce an axis, OX.

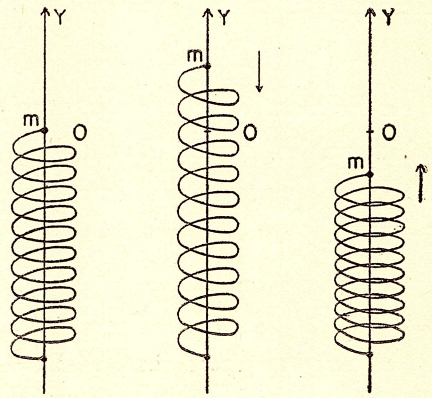

Fig. 8

If $y(t)$ denotes the coördinate of the particle at the moment, t, then its velocity is $\frac{dy}{dt}$ or $y'(t)$, its acceleration is $\frac{d^2y}{dt^2}$ or $y''(t)$. By one of Newton's laws, the force producing an acceleration of a mass is equal to mass times acceleration—in our case, is equal to $my''(t)$, if distance, time, and mass are measured in centimeters, seconds, and grams.

The force acting on our particle is the tendency of the spring to return to the neutral position when it is extended or compressed. By Hooke's law, this force, called *elasticity of the spring*, is proportional to the displacement of the end of the spring. Thus, if we call H Hooke's constant of proportionality, we have for each t

$$m\,y''(t)=H\,y(t).$$

The acceleration due to the elasticity of the spring tends to decrease $y(t)$ when $y(t)>0$, and to increase $y(t)$ when $y(t)<0$ (Fig. 8). Thus, H is negative, and $-H$ is positive. We set $-H=h$. The function, $y(t)$, satisfies the differential equation,

$$m\,y''(t)+h\,y(t)=0 \text{ or } y''(t)+\frac{h}{m}y(t)=0, \qquad \textbf{XXXIV}$$

which is of the type of XXXIIb. Since the mass, m, is greater than 0, we have $\frac{h}{m}>0$. Hence the solutions of XXXIV read

$$y(t)=A\cos\sqrt{\frac{h}{m}}t+B\sin\sqrt{\frac{h}{m}}t.$$

If at $t=0$ the particle passes through the neutral position, then $y(0)=0$.

Thus, $A \cos 0 + B \sin 0 = 0$, and consequently, $A = 0$. The solutions are

$$y(t) = B \sin \sqrt{\frac{h}{m}}\,t$$

for some constant, B. The velocity of the particle is

$$y'(t) = B\sqrt{\frac{h}{m}} \cos \sqrt{\frac{h}{m}}\,t.$$

If at $t=0$ the particle has the velocity, v_0, then

$$v_0 = B\sqrt{\frac{h}{m}} \text{ and } B = v_0\sqrt{\frac{m}{h}}.$$

Only one solution of XXXIV satisfies the condition that at $t=0$ the particle passes through the neutral position and has the velocity, v_0—*viz.*, the solution,

$$y(t) = \sqrt{\frac{m}{h}}\,v_0 \sin \sqrt{\frac{h}{m}}\,t.$$

We obtain an application to electricity by studying a circuit containing in series

- **a** an inductance of L henries $(L \neq 0)$
- **b** a resistance of R ohms
- **c** a source of e.m.f. of $E(t)$ volts at the moment, t
- **d** a condenser whose capacity is C farads.

If we call $V(t)$ the potential difference of the plates of the condenser at the moment, t, the current, $i(t)$, satisfies the differential equation,

$$L\frac{di}{dt} + Ri + V = E;$$

that is,

$$Li'(t) + Ri(t) + V(t) = E(t) \text{ for each } t.$$

Now, if we call $Q(t)$ the charge on a condenser plate at the moment, t, then

$$Q(t) = CV(t) \text{ and } i(t) = \frac{dQ(t)}{dt} = Q'(t).$$

Hence, $Q(t)$ satisfies the second order differential equation,

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t). \qquad \text{XXXV}$$

In case $E(t)=0$ and $R=0$ (that is, no e.m.f. is impressed and the resistance is negligible), we have $LQ''(t) + \frac{1}{C}Q(t) = 0$ and this equation can be solved like that of the spring.

TEST YOUR ABILITY TO APPLY SECOND ORDER DIFFERENTIAL EQUATIONS

10 Find the solution of XXXIV describing a particle which at $t=0$ passes through the point with the coördinate, y_0, and has the velocity, 0.

11 Find the solution of XXXV in case $E(t)=0$ and $R=0$, if $L=5$ henries and $C=5 \cdot 10^{-6}$ farads.

VIBRATIONS IN A RESISTING MEDIUM

We now solve the differential equation,

$$y''(t)+ay'(t)+by(t)=0 \qquad \text{XXXVI}$$

where a and b are constants. Instead of looking for the function, $y(t)$, we try to find an auxiliary function,

$$z(t)=e^{\frac{a}{2}t}\cdot y(t).$$

From this definition, it follows that

$$y(t)=e^{-\frac{a}{2}t}\cdot z(t),$$

$$y'(t)=-\frac{a}{2}e^{-\frac{a}{2}t}\cdot z(t)+e^{-\frac{a}{2}t}\cdot z'(t).$$

$$y''(t)=\frac{a^2}{4}e^{-\frac{a}{2}t}\cdot z(t)-ae^{-\frac{a}{2}t}\cdot z'(t)+e^{-\frac{a}{2}t}\cdot z''(t).$$

The left side of equation XXXVI reads

$$y''(t)+ay'(t)+by(t)=e^{-\frac{a}{2}t}\cdot\left[z''(t)+\left(b-\frac{a^2}{4}\right)z(t)\right].$$

If XXXVI holds, the product in the expression on the right side of the last equality equals 0 for each t. The first factor, $e^{-\frac{a}{2}t}$, cannot equal 0 for any t. Consequently, the expression in the brackets equals 0; that is to say, the auxiliary function, $z(t)$, satisfies the equation,

$$z''(t)+\left(b-\frac{a^2}{4}\right)z(t)=0.$$

This equation is of the type of XXXIIb, the constant, $b-\frac{a^2}{4}$, playing the rôle of b. The nature of the solutions depends upon the sign of this constant. It is a trigonometric function if $b-\frac{a^2}{4}>0$; it is an exponential function if $b-\frac{a^2}{4}<0$; it is a linear function if $b-\frac{a^2}{4}=0$. Multiplying the auxiliary function, $z(t)$, by $e^{-\frac{a}{2}t}$, we obtain for the function, $y(t)$, the following expressions:

$$y(t)=e^{-\frac{a}{2}t}\left(A\cos\sqrt{b-\frac{a^2}{4}}\,t+B\sin\sqrt{b-\frac{a^2}{4}}\,t\right) \quad \text{if } b-\frac{a^2}{4}>0$$

$$y(t)=e^{-\frac{a}{2}t}\left(Ae^{\sqrt{\frac{a^2}{4}-b}\,t}+Be^{-\sqrt{\frac{a^2}{4}-b}\,t}\right) \quad \text{if } b-\frac{a^2}{4}<0$$

$$y(t)=e^{-\frac{a}{2}t}\,(A+Bt) \quad \text{if } b-\frac{a^2}{4}=0$$

Substituting these functions into XXXVI, we find that they actually are solutions.

Plotting functions

It is important for the reader to understand the nature of these solutions.

The best way of getting an insight is to plot the functions, $y(t)$ versus t. We shall plot those solutions which satisfy the conditions, $y(0)=0$ and $y'(0)=1$. We shall therefore obtain curves passing through the origin and having there the slope, 1; thus, a tangent with an inclination of 45°. The reader will easily verify that we have to plot the function,

$$y(t)=\frac{1}{\sqrt{b-\frac{a^2}{4}}}\,e^{-\frac{a}{2}t}\sin\sqrt{b-\frac{a^2}{4}}\,t \qquad \text{if } \frac{a^2}{4}<b$$

$$y(t)=t\,e^{-\frac{a}{2}t} \qquad \text{if } \frac{a^2}{4}=b$$

$$y(t)=\frac{1}{2\sqrt{\frac{a^2}{4}-b}}\,e^{-\frac{a}{2}t}\left(e^{\sqrt{\frac{a^2}{4}-b}\,t}-e^{-\sqrt{\frac{a^2}{4}-b}\,t}\right) \qquad \text{if } \frac{a^2}{4}>b.$$

We choose $b=1$ and plot these curves for various values of a. For $a=0$, we obtain the ordinary sine curve, $y=\sin t$ of Fig. 9a, consisting of waves of period 2π and frequency $\frac{1}{2\pi}$ and constant amplitude 1. For small a, we obtain a curve represented in Fig. 9b, consisting of waves of slightly larger period than 2π; *viz.*, of period $\frac{2\pi}{\sqrt{1-\frac{a^2}{4}}}$ and frequency $\frac{1}{2\pi}\sqrt{1-\frac{a^2}{4}}$, whose amplitudes gradually decrease. For a larger a (still <2), we arrive at a curve like that represented in Fig. 9c, whose period is still larger and whose frequency still smaller than that of the preceding curve, while the amplitudes fade out still more rapidly. It is interesting to compute the value, t_1, of t for which the first maximum of $y(t)$ occurs. It turns out to be

$$t_1=\frac{1}{\sqrt{1-\frac{a^2}{4}}}\arctan\frac{2}{a}\sqrt{1-\frac{a^2}{4}}.$$

As a approaches 2, we find that t_1 approaches 1. For $a=2$, we arrive at the function, te^{-t}, which increases from $t=0$ to $t=1$, then decreases but without ever reaching 0. Thus, the curve of Fig. 9d does not present any waves

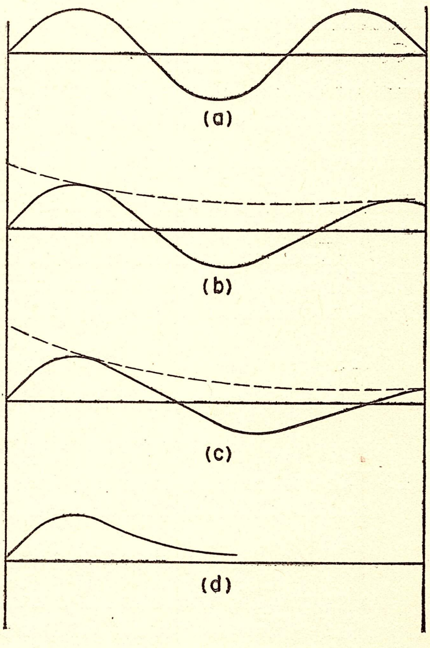

Fig. 9

and does not exhibit any periodicity. The same is true for the curve representing the case that $\frac{a^2}{4} > b$.

First, we apply equation XXXVI to a mechanical problem. A medium resists the motion of a particle with a force proportional to the velocity of the particle. If a particle moves along the Y-axis, and its ordinate at the moment, t, is $y(t)$, then its velocity is $y'(t)$. The resistance of the medium is $R \cdot y'(t)$. The constant, R, is negative. We set $-R = r$. The number, r, is small in air, much larger in water, and still much larger in a viscous liquid or mercury.

On page 532, we saw that the coördinate of a particle of mass, m, fixed to the end of an elastic spring satisfied the equation,

$$m\,y''(t) = H\,y(t).$$

If the motion takes place in a resisting medium, then the equation reads

$$m\,y''(t) = H\,y(t) + R\,y'(t) \text{ or } m\,y''(t) + r\,y'(t) + h\,y(t) = 0. \qquad \text{XXXVII}$$

Dividing by m, we obtain an equation of the type of XXXVI with $a = \frac{r}{m}$ and $b = \frac{h}{m}$. The nature of the solutions depends upon the sign of the number,

$$b - \frac{a^2}{4} = \frac{h}{m} - \frac{r^2}{4m^2} = \frac{1}{m}\left(h - \frac{r^2}{4m}\right).$$

In air, $\frac{r^2}{4m}$ is small compared with h, and the curve, $y = y(t)$, looks like that of Fig. 9b. In water, $\frac{r^2}{4m}$ is much larger than it is in air but still smaller than h if the spring is strong. The curve, $y = y(t)$, looks like that of Fig. 9c. The number of oscillations in a given period of time is smaller than it is in air; that is to say, in water the vibration is slower than in air. Moreover, it fades out more rapidly than in air. In mercury, $\frac{r_2}{4m} > h$. The curve, $y = y(t)$, looks like that in Fig. 9d. The particle at the end of the spring does not oscillate at all. It slowly approaches the neutral position.

We obtain an application to an electrical problem by considering equation XXXV in case $E(t) = 0$. It describes a circuit with inductance, L, resistance, R, and a condenser of capacity, C. Dividing by L, it reads

$$i''(T) + \frac{R}{L} i'(t) + \frac{1}{LC} i(t) = 0,$$

which is of the type of XXXVI.

TEST YOUR ABILITY TO PLOT FUNCTIONS

12 The reader may treat the cases where $L = 5$ henries, $C = 4 \cdot 10^{-6}$ farads, and $R = 100$ ohms, 1000 ohms, 3000 ohms.

RESONANCE PHENOMENA

On page 524, we found by trial and error one solution of the equation, $Li' + Ri = E \cos \alpha t$, and we saw that any solution could be obtained by adding a solution of the equation, $Li' + Ri = 0$. In a similar way, we shall now solve the equation,

$$y''(t) + ay'(t) + by(t) = M \cos \alpha t. \qquad \text{XXXVIII}$$

We try

$$y(t)=N\cos(\alpha t+\varphi).$$

For this function, we have

$$y'(t)=-N\alpha\sin(\alpha t+\varphi)$$

and

$$y''(t)=N\alpha^2\cos(\alpha t+\varphi).$$

Hence,

$$y''(t)+ay'(t)+by(t)=(N\alpha^2-bN)\cos(\alpha t+\varphi)-aN\alpha\sin(\alpha t+\varphi).$$

If $y(t)$ is to be a solution of XXXVIII, we must have

$$(N\alpha^2-bN)\cos(\alpha t+\varphi)-Na\alpha\sin(\alpha t+\varphi)=M\cos\alpha t \text{ for each } t.$$

For $t=\frac{\pi}{2}$, we obtain

$$(\alpha^2-b)\cos\left(\frac{\pi}{2}+\varphi\right)-a\alpha\sin\left(\frac{\pi}{2}+\varphi\right)=0;$$

thus,

$$\tan\left(\frac{\pi}{2}+\varphi\right)=\frac{\alpha^2-b}{a\alpha}$$

and hence,

$$\tan\varphi=\frac{a\alpha}{b-\alpha^2} \text{ and } \varphi=\text{arc tan}\frac{a\alpha}{b-\alpha^2}.$$

For $t=-\frac{\varphi}{\alpha}$, we obtain

$$N(\alpha^2-b)=M\cos(-\varphi)=M\cos\varphi.$$

Now

$$M\cos\varphi=M\frac{1}{\sqrt{1+\tan^2\varphi}}=M\frac{b-\alpha^2}{\sqrt{(b-\alpha^2)^2+a^2\alpha^2}}.$$

Thus, if $\alpha^2-b\neq 0$, we have

$$N=-\frac{M}{\sqrt{(b-\alpha^2)^2+a^2\alpha^2}}.$$

By substitution, we find that the function,

$$y(t)=-\frac{M}{\sqrt{(b-\alpha^2)^2+a^2\alpha^2}}\cos\left(\alpha t+\text{arc tan}\frac{a\alpha}{b-\alpha^2}\right)$$

actually is a solution of XXXVIII. Any solution of XXXVIII is of the form obtained by adding a solution of equation XXXVI.

These results form the basis of our understanding of the so-called resonance phenomena, which account for the collapse of bridges and houses on one hand and are used in radio engineering and for the tuning of musical instruments on the other. On page 532, we saw that the coördinate at the moment, t, of a particle of mass, m, at the end of an elastic spring in a resisting medium satisfies the equation,

$$my''+ry'+hy=0.$$

If r is small, then the particle oscillates about the neutral position. Gradually the motion will fade out. Now let the particle be subjected to some external force of periodically varying intensity. If $F\cos\alpha t$ is the intensity

of the force at the moment t, then the coördinate, $y(t)$, of the particle satisfies the differential equation,

$$my''(t)+ry'(t)+hy(t)=F\cos\alpha t$$

or

$$y''+\frac{r}{m}y'+\frac{h}{m}y=\frac{F}{m}\cos\alpha t,$$

which is of the type of equation XXXVII. The solutions of this equation are obtained by adding a solution of equation XXXVIII $\left(y''+\frac{r}{m}y'+\frac{h}{m}y=0\right)$ to a function,

$$y(t)=-\frac{F}{m\sqrt{\left(\frac{h}{m}-\alpha^2\right)^2+\frac{r^2}{m^2}\alpha^2}}\cos(\alpha t+\arc\tan\varphi). \qquad \text{XXXIX}$$

The particle carries out two motions: the oscillation of frequency, $\frac{2\pi}{\sqrt{\frac{h}{m}-\frac{r^2}{4m^2}}}$, which fades out and the oscillation (XXXIX) of frequency $\frac{2\pi}{\alpha}$, which is steady. After some time, the steady part is the only important one. Very interesting is the intensity of the steady vibration, XXXIX. If the denominator on the right side of XXXIX is small, then the steady vibration has a large amplitude. The denominator is small if the resistance, r, is small and $\frac{h}{m}-\alpha^2$ is small—that is to say, the frequency of the impressed force is close to the frequency of the solution of the equation,

$$y''+\frac{h}{m}y=0,$$

describing the particle at the end of the same spring without any resisting medium and without impressed forces. We see that, if the resistance of the medium is small and the frequency of the impressed force is that of the spring, then a small external force produces large oscillations. They are called *sympathetic vibrations*.

A well-known example in the field of acoustics is the following phenomenon.

If we strike a tuning fork of pitch A, the end vibrates with a frequency of 440 per second. This oscillation of the tuning fork is transmitted to the surrounding air, producing a motion so weak that no part of our skin would perceive it except the ear drum, which is especially adapted to the perception of sound waves.

If the waves of the air strike tuning forks of pitch G vibrating with a frequency of 392 per second, or even G-sharp or B-flat, they do not affect these forks at all. However, if the weak air waves of frequency 440 happen to strike another tuning fork of pitch 440, then there will be a marked effect.

Even if the second tuning fork is completely at rest, it will start vibrating (a "sympathetic vibration") as though it had been struck with an appreciable force. The explanation is found in the preceding results. An end point of the second tuning fork will carry out a vibration of frequency 440 if the fork is struck.

If a periodic external force of this frequency is impressed on the tuning

fork, then the end will start vibrating with an amplitude large enough to make the vibration audible and even visible, even if the intensity of the external force is as weak as that of the vibrating air.

PARTIAL DIFFERENTIAL EQUATIONS

The differential equations studied in the preceding sections deal with functions of one variable and its derivatives. Such equations are called ordinary differential equations. If we ask for a function, $y(x,t)$, of two variables whose partial derivatives, $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$, satisfy a certain condition, then we try to solve what is called a partial differential equation.

A simple example of a partial differential equation is

$$\frac{\partial y}{\partial x}=x\text{—that is, }\frac{\partial y(x,t)}{\partial x}=x\text{ for each }x\text{ and }t.$$

One easily verifies that $y(x,t)=\frac{x^2}{2}$ is a solution. If $f(t)$ is any function of t, then $\frac{x^2}{2}+f(t)$ likewise is a solution, for

$$\frac{\partial}{\partial x}\left[\frac{x^2}{2}+f(t)\right]=x.$$

Another example is

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial t}\text{—that is, }\frac{\partial y(x,t)}{\partial x}=\frac{\partial y(x,t)}{\partial t}\text{ for each }x\text{ and }t. \qquad \textbf{XL}$$

$x+t$ is a solution; $(x+t)^2$ is another solution; $\sin(x+t)$ still another. In general, we see that, if $f(z)$ is any function of one variable, then $f(x+t)$ is a solution of our partial differential equation.

Even an ordinary differential equation has many solutions. We saw that, for each constant, C, the function, $y(x)=Ce^x$, was a solution of the equation, $y'(x)=y(x)$. For any pair of numbers, A and B, $y(x)=A\cos 3x+B\sin 3x$ is a solution of the second order differential equation, $y''(x)=9y(x)$. The totality of solutions of partial differential equations is still larger. Not only is $C(x+t)$ a solution of XL but, in addition, for each function, $f(z)$, the function, $f(x+t)$, is also a solution of the equation, $\frac{\partial y}{\partial x}=\frac{\partial y}{\partial t}$.

Two examples will illustrate the great importance of partial differential equations, and the fact that they admit so many solutions. Let the end points of an elastic string with length of 1 be fixed at the points, (0,0) and (1,0), on the X-axis. Let us assume that the string is plucked in such a way that all its points remain in the XY-plane and that each point of the string moves parallel to the Y-axis and thus throughout its motion has the same abscissa. Each point will tend to return to its neutral position on the X-axis. We call $y(x,t)$ the ordinate at the moment, t, of the particle of the string whose abscissa is x. Then one can show that

$$\frac{\partial^2 y}{\partial t^2}=k\frac{\partial^2 y}{\partial x^2}\text{—that is, }\frac{\partial^2 y(x,t)}{\partial t^2}=k\frac{\partial^2 y(x,t)}{\partial x^2}\text{ for each }x\text{ and }t. \qquad \textbf{XLI}$$

This partial differential equation expresses the fact that the acceleration, $\frac{\partial^2 y(x,t)}{\partial t^2}$, of any particle and at any moment is proportional to $\frac{\partial^2 y(x,t)}{\partial x^2}$. The last expression is essentially the curvature of the string at that particle and at that moment. The fact that equation XLI has so many solutions makes it possible to determine $y(x,t)$—that is, the position of each particle at each moment, no matter where the particles were at the moment, $t=0$—that is to say, no matter what the initial shape of the string was when it was plucked.

Another partial differential equation of the second order describes the current, $i(x,t)$, at the point with abscissa, x, and at the moment, t, in a wire along the X-axis. Let the resistance of the cable be r ohms per mile, its capacity C farads per mile. It can be shown that

$$\frac{\partial^2 i}{\partial x^2}=rC\frac{\partial i}{\partial t}.$$

Knowing the current at the time, $t=0$, we can determine the function, $i(x,t)$ by solving the partial differential equation.

· MENSURATION ·

By Robert Baker, B.S.

WHENEVER we take a measure of something, we are compelled to express our answer in terms of something definitely known by our associates so that they will know what we wish to convey. This has given rise to the setting of standards of measurement. The standards for the fundamental dimensional quantities must be purely a matter of definition. In years gone by, when requirements for measurements were relatively inexact, crude standards were set. The setting of new standards and systems of units in the past has led to serious confusion today, especially in regard to the determination of mass.

A *primary standard* is one which is set by definition, such as the platinum-iridium bar kept in France with a length marked thereon known as the international meter. A *secondary standard* duplicates as nearly as is scientifically possible the quantity set by a primary standard. Secondary standards of the international meter are kept by the Bureau of Standards in Washington. Meter sticks, like yard sticks, are made to conform to secondary standards within reasonable practical limits, and are often referred to as *tertiary standards*. The various standards bearing on subjects in this article will be dealt with in their place.

MEASUREMENT OF LENGTHS AND ANGLES

All methods of measuring length consist essentially of comparing the length of the object to be measured with a calibrated standard of length, which is used as the norm. Some measurements must be made more precisely than others.

Non-precision instruments

We shall first consider measurements where the precision required is not high (where units smaller than one-hundredth of an inch are not considered); that is, so-called non-precision measurements. The calibrated standard employed in such work consists generally of a strip of metal with lines ruled along one edge at a right angle to the edge, the lines being a known distance apart. This known distance is called the interval of graduation, and is rarely less than 0.01″. If the standard is designed to measure distances less than four or five feet, it is usually made of a fairly rigid strip, and is known as a *rule*, or *scale*. If it is designed to measure larger distances, it is made of a flexible strip which may be rolled into a compact form, and is known as a *tape*. The interval of graduation of tapes varies with their use and may be one-sixteenth of an inch, one-tenth of an inch, or one-hundredth of a foot.

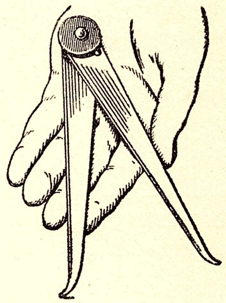

Fig. 10

Sometimes it is impossible to bring the scale into contact with the object to be measured. It is then necessary that some intermediate device be used to transfer the unknown distance to the scale. *Calipers* are used for this purpose. Two sorts of calipers are shown in Figs. 10 and 11, one for measuring internal dimensions, the other for external dimensions. When they are in use, the distance between the tips of the calipers is adjusted until it is equal to the dimension to be measured. The distance is then read by comparing the calipers with a scale.

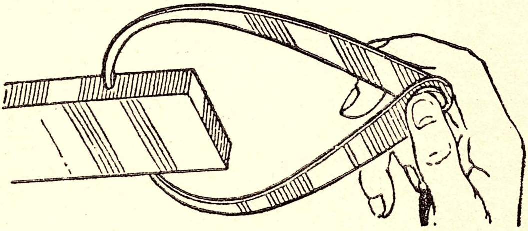

Fig. 11

The measurement of angles, like the measurement of distances, is usually accomplished by methods of comparison. The standard in this case is known as the *protractor*. Measurement of angles was discussed in our study of trigonometry, pages 385 to 392.

Precision instruments

As has been stated, the interval of graduation of a scale is seldom smaller than one one-hundredth of an inch (0.01″); hence, it is apparent that, if one end of an unknown distance is placed opposite a scale graduation, the chances are that the other end will fall between two graduations. In non-precision work, the graduation which falls nearer the end is read, yet it is often desirable to read the distance more accurately. Other means than the ordinary scale must be employed in making so-called "precision measurements".

The heart of many precision measuring instruments is the *vernier.* It is a small supplementary scale which enables the user to read with certainty a value which he would otherwise have to estimate on the main scale. One instrument which employs a vernier scale is the vernier caliper, shown in Fig. 12. The small scale engraved on the movable jaw of the instrument is the vernier scale, and is so placed that, when the jaws are closed, the zero line of the vernier scale coincides with the zero line of the main scale. In use, the jaws are opened to the desired width and clamped. The reading of both scales indicates the spacing between the jaws and hence the dimension. The first reading is taken on the main scale and is that division which falls directly opposite, or most closely precedes, the vernier zero line.

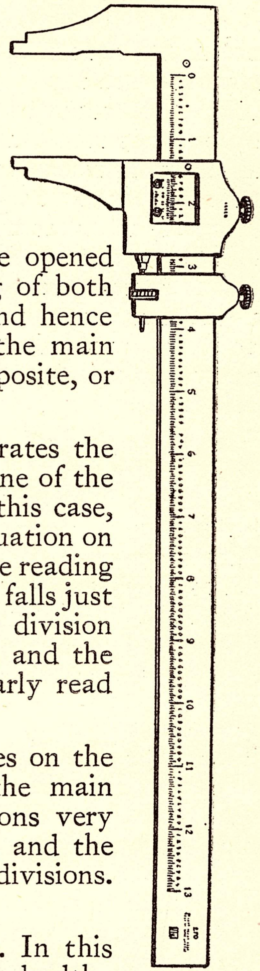

Fig. 12

A decimal main scale with a vernier demonstrates the layout of the vernier scale. In Fig. 13, the zero line of the vernier falls at 1.3 on the main scale. Since, in this case, the vernier zero-mark lines up exactly with a graduation on the main scale, the second decimal place is 0 and the reading is 1.30. In Fig. 14, the zero line of the vernier scale falls just beyond 1.3 on the main scale. Here the vernier division 1, lines up with a graduation on the main scale, and the reading is 1.31. In Fig. 15, the setting is similarly read as 1.36.

Fig. 16 is a special case where none of the lines on the vernier corresponds with the graduations on the main scale. In this case, two adjacent vernier divisions very nearly correspond with two main-scale divisions, and the correct reading falls between these two vernier divisions. Here the reading is 1.365.

The explanation of the vernier is not difficult. In this case, one vernier interval equals nine one-hundredths (0.09) of an inch, whereas one main-scale division equals ten one-hundredths (0.10) of an inch. Therefore, if the

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The vernier scales shown in Figs. 13, 14, 15, and 16 are not designed to represent an instrument as they are sketched, but are simplified in detail so that the layout and reading of the vernier will be apparent. Practice in handling a vernier instrument will quickly demonstrate the extraordinary value of this device.

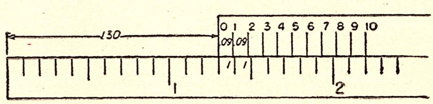


Fig. 13

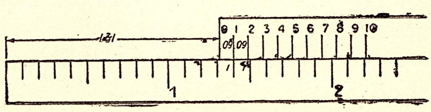


Fig. 14

vernier zero (0) is opposite a main-scale division, the vernier 1 is 0.01″ from the next main-scale division (Fig. 13). If the vernier scale is moved 0.01″ (Fig. 14), the vernier division, 1, lines up with the next main-scale division. Similarly, if the vernier scale is moved another 0.01″ (or a total of 0.02″), the vernier division 2 lines up with the following main-scale division. Therefore, if the vernier 6, or any other division, lines up with a division on the main scale, it indicates that the vernier, 0, has moved that many one-hundredths from the original main-scale division. Thus, the actual dimension is shown on the main scale by the position of the vernier 0, but the precise reading to hundredths of an inch must be taken from the vernier scale.

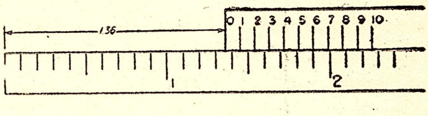


Fig. 15

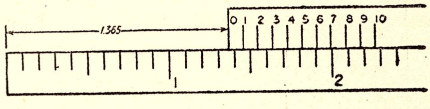


Fig. 16

The main scale of the vernier caliper just described was graduated in one-hundredths of an inch. This makes for ease in explaining the reading of the instrument, but in use such coarse graduations may be inadequate. The main scales of most vernier calipers are graduated to 0.025″, and the vernier scales have 25 divisions, which equal 24 divisions of the main scale. The principle of reading is exactly the same as described for the decimal scale.

The *least count* of a vernier is the change in measurement which will cause a shift of one division on the vernier scale. In the cases illustrated, it is 0.01″. In the common vernier caliper described above, where each main-scale division is 0.025″, the least count is 0.001″. In some cases, it is $\frac{1}{128}''$. In the case of the vernier protractor (described on page 547), it is often 5 minutes, or $\frac{1}{12}$ of one degree.

Another precision measuring instrument which is in universal use is the *micrometer* (Fig. 18). The spacing between the jaws may seldom be varied more than an inch on any single micrometer because the micrometer head would be long and unwieldy. The figure shows a micrometer which can measure an object between 0″ and 0.5″. An assortment of frames is often supplied, with one interchangeable head; the smallest frame is used to measure dimensions from 0″ to 1″, the next from 1″ to 2″, the next from 2″ to 3″, and so on. This is where the vernier caliper shows to good advantage, for the spacing between its jaws depends only upon the length of the main scale, which is seldom less than six inches. Hence, six micrometer frames of different sizes would be required to cover a range of measurement for which one vernier caliper would be sufficient.

The micrometer is built about a screw thread (generally of forty threads per inch) very accurately machined on a steel spindle. The spindle is moved in and out by the revolution of the screw inside the nut on the micrometer frame. A hollow barrel is part of the micrometer frame and holds the nut firmly in place. As the spindle moves inside the nut, the thimble moves outside the barrel. The barrel carries a scale divided into tenths, with four subdivisions for each tenth. One turn of the thimble advances the spindle one-fortieth of an inch (0.025″) along the barrel. It is evident that, if the thimble turns through a fraction of a revolution, the spindle will advance the same fraction of one-fortieth of an inch. A graduated scale is etched on the edge of the thimble, dividing the circumference into 25 parts. Hence, turning the thimble one division $\left(\frac{1}{25}\text{ revolution}\right)$ advances the spindle $\left(\frac{1}{25}\times 0.025=0.001''\right)$ one one-thousandth of an inch. To enable accurate reading to the fourth decimal place (ten-thousandths), a vernier scale is often ruled on the barrel to be used in conjunc-

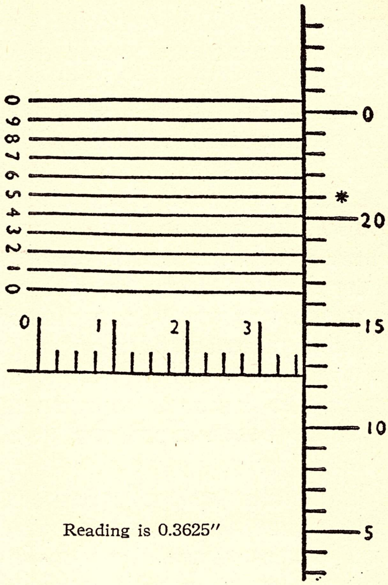


Fig. 17

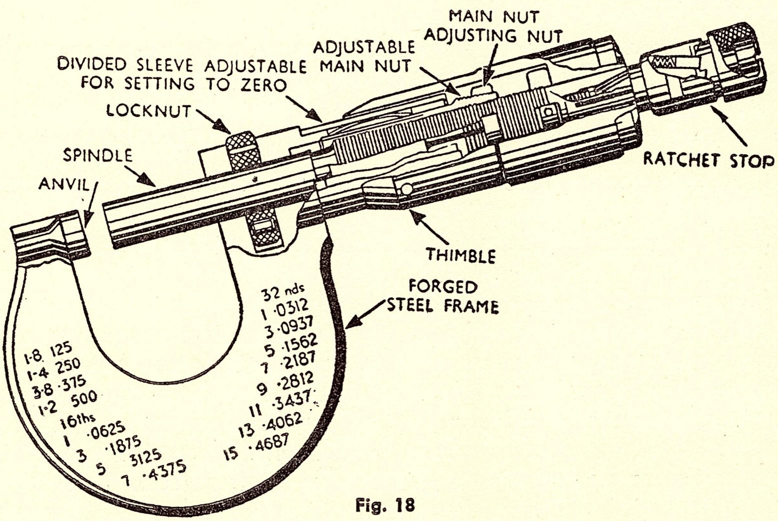


Fig. 18

tion with the thimble scale. The instrument is then known as the *vernier micrometer.* Fig. 17 shows the scale of such an instrument.

Micrometers can be obtained for measuring in the metric system. These have a screw or spindle with 20 threads per cm, so that one complete revolution of the thimble moves the spindle 0.5 mm. Apart from this, the form of construction and method of use are the same as for an instrument calibrated in fractions of an inch.

Another precision device for the measurement of length, and one which has a very broad application, is the *dial gage* (Fig. 19). It consists of a dial and a pointer which is connected by a magnifying linkage with a sliding-rod projecting from the casing of the instrument. The dial gage is arranged in such a way that a definite displacement of the rod causes the pointer to make one revolution. Commercial dial gages read to one thousandth or one ten-thousandth of an inch. Dial gages are seldom used to find over-all dimensions, but find wide use in inspection and control work, such as gaging the thickness of metal sheeting.

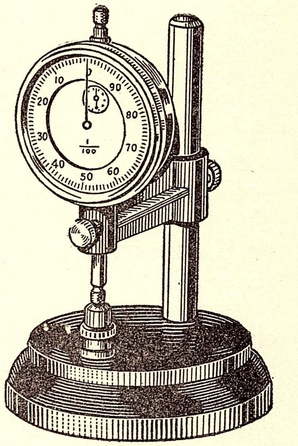


Fig. 19

GAGE BLOCKS

The most accurate measuring standards in common use are *gage blocks.* Gage blocks are metal blocks, rectangular in all sections, which have been machined and polished until their dimensions between two opposite faces at standard temperature conditions are accurate to a few millionths of an inch.

Johansson gage blocks are made in three grades, the finest grade (AA) are accurate to ± two one-millionths of an inch per inch of block length; the second grade (A) are accurate to ± four one-millionths of an inch per inch; and the third grade (B) are accurate to ± eight one-millionths of an inch per inch. The AA grade is used only to make an occasional check of the accuracy of the A-grade blocks, which in turn are used to keep frequent check on the B-grade, or working grade, blocks. The B-grade blocks are used to check other measuring instruments, or to check the dimensions of finished work when the tolerance is small. Hoke gage blocks are made in two grades, the higher, or "laboratory grade", being similar to the Johansson "AA" blocks, and the lower being accurate to ± five one-millionths of an inch per inch. The blocks are made in various sizes and arranged in sets so that common fractional dimensions may be equalled to the nearest ten-thousandths by some combination of blocks.

Gages are also made in other forms, *ring gages* and *plug gages* being

the most common. These generally are slightly less precise than Johansson or Hoke gage blocks, and are accurate, in working grades, to about one ten-thousandth of an inch per inch.

Gage blocks represent the most precise mechanical means of measuring distance that has yet been devised. It is obvious, then, that some non-mechanical means must be employed in the checking of master gage blocks. Optical methods are employed for this and other super-precision work. The apparatus required for the checking of gage blocks consists of a level surface plate, a source of light*, and an *optical flat*. An optical flat is a piece of clear glass with parallel sides which do not vary by more than 0.000002″ from flatness. When the optical flat is placed on the gage block under the light, dark lines appear on the block's surface if it is not absolutely flat. The fewer the bands which appear, the nearer the block is to being a flat or plane surface. (To explain the theory of these "interference bands" would go beyond the scope of this work. The reader is referred to the chapters on the interference of light in physics textbooks.)

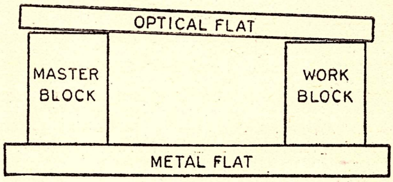


Fig. 20

The optical flat is useful also in checking the length of a working gage block with that of a standard master-block by standing the blocks on end on a surface plate and placing the flat across the tops of the two blocks (Fig. 20). The greater the number of interference bands appearing above the master-block, the greater the difference in block length.

TEST YOUR ABILITY TO USE GAGE BLOCKS

A certain set of gage blocks contains blocks of the following sizes:

0.1001″ to 0.1009″ in steps of 0.0001″
0.1010″ to 0.1090″ in steps of 0.0010″
0.1100″ to 0.1900″ in steps of 0.0100″
0.1000″ to 0.5000″ in steps of 0.1000″
1.0000″ to 4.0000″ in steps of 1.0000″

What arrangement of blocks should be chosen to equal, in the smallest number of blocks, the following dimensions to the nearest ten-thousandth?

1 0.4679″ **2** $\frac{119''}{128}$ **3** $9\frac{5}{16}''$

(*Hint*: Begin with the last digit of each dimension.)

Another optical measuring device which is capable of high precision work is the *measuring microscope*. It consists of a microscope with

* This light must be monochromatic, which, as the name implies, is "one-colored light", rather than multi-colored light, such as daylight or common artificial light.

a scale of high precision mounted inside the instrument in such a way that the scale magnified and the object to be measured both appear in focus. The length of image is read with the aid of hairlines which are moved by a micrometer screw.

The precision measurement of angles is achieved by the use of a very accurately ruled protractor with a vernier scale attached at the periphery in such a way that it may be moved to any position. It is known as a *vernier protractor*. The center of the protractor circle is placed over the vertex of the angle to be measured, the zero line of the protractor is brought in line with a side of the angle, and the vernier is moved around to the position where the zero line of the vernier coincides with the other side of the angle.

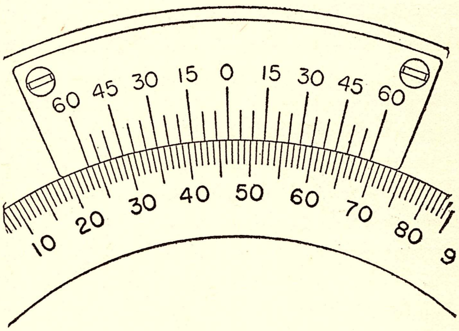


Fig. 20a

Accurate vernier protractors are to be found in surveyor's *transits* and *theodolites*. A transit is an instrument for the precision measurement of angles on the earth's surface, and a theodolite is a particularly fine and carefully constructed transit. Both instruments consist essentially of a mounted telescope with internal cross-hairs. A circular, horizontal plate (the "upper circle") is fastened rigidly to the telescope and mounted on a vertical spindle around which the telescope and plate are free to revolve as a unit. The upper circle usually carries the vernier scale. A second horizontal plate (the "lower circle"), larger than the upper circle, carries the main scale. The reading of the vernier and protractor scales determines the position of the telescope tube with respect to the lower circle. During all measurements, the spindle about which the telescope rotates must be in a vertical position. To assist in accomplishing this, level bubbles are provided, and the instrument is "levelled up" before sighting.

When the transit is in use, it is placed with the aid of a plumb-bob, over the vertex of the angle to be measured. The telescope is then turned until it is in line with one of the points which is being used to determine the angle. The position of the telescope, as read on the lower circle and vernier, is recorded. The telescope is rotated until it is in line with the other point used to determine the angle. When the sighting point and cross-hairs have been lined up carefully, the position indicated by the protractor and vernier scale is again recorded. The difference between the two recorded settings is the angle being measured.

Most transits have a least count of ten seconds, and a few will read as close as five seconds of angle.

MEASURING INSTRUMENTS OF AREAS AND VOLUMES

A plane area is a measure of the amount of surface within a closed boundary. The unit of area in any system of measurement is the surface of a plane square having sides of one unit-length in that system. The side of the unit square may be one inch, one foot, one centimeter, one meter, etc., and the unit of area would then be the square inch, the square foot, the square centimeter, or the square meter, often written in.^{2}, ft.^{2}, cm.^{2}, m.^{2}. All of the surface under consideration may or may not lie in one plane. For example, the surface inside the boundary of a triangle lies on one plane, while the surface of a sphere does not. The areas of mathematically determined surfaces may be computed from formulae derived by plane or solid geometry and calculus, as has been shown. Since there are no simple devices for measuring the areas of solids or space-curved figures, mathematical methods using indirect measurements usually are employed.

Areas

The very definition of the unit of area suggests a method of measuring an area of an irregular figure. Some methods for approximating the correct measurements of irregular areas were presented on pages 313 and 314.

The *Amsler Polar Planimeter* is a very convenient and accurate instrument for the direct measurement of areas (Fig. 21). A tracing arm is pivoted to a radial arm at the point, P. The pivot point, P, moves in a circular arc about the pin-point center, F. Any point on the boundary is chosen for a terminal; the tracing point is set there, and the initial planimeter reading is read on the graduated wheel and vernier. The tracing point, T, is run once over the boundary in a clockwise direction to the terminal point, when the final reading is taken. The final reading minus the initial is the desired area within the boundary.

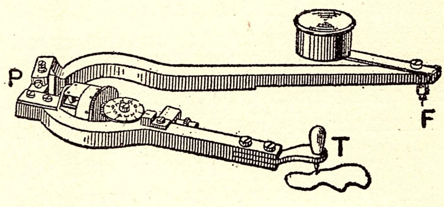


Fig. 21

The above discussion of the planimeter applies to the simple case as sketched, where the area is relatively small and the fixed point is *outside* the boundary. When the area is so large that the fixed point falls *inside* the boundary, the area read must be either added to or subtracted from the *zero circle area*. The zero circle area is usually stamped on the planimeter. If the tracing arm makes an angle on the average with the radial arm of more than 90°, the area is added

to the zero circle area; if the angle is on the average less than 90°, the area is subtracted from the zero circle area.

The *Coffin averager* is a special planimeter used mainly to determine in a single operation the mean ordinate $\left(\frac{\text{area}}{\text{length}}\right)$ of an irregular area. This is especially well adapted to finding the mean effective pressure of an indicator card in engine testing.

Volumes

The unit of volume in any system is the volume of a cube having sides with a length of one unit in that system. Common units of volume are the cubic inch, cubic foot, cubic centimeter, and cubic meter, often written as in.^{3}, ft.^{3}, cm.^{3} (or c.c.), and m.^{3}. In the metric system, 1000 cubic centimeters comprise a liter, which is in very common laboratory use. A liter is 1.06 quarts. There are one thousand liters in a cubic meter.

Dry measures, such as pecks and bushels, are of such common knowledge that we shall not dwell on them here except to give them the appropriate place in the table.

Wet measurements in small quantities are made, almost invariably, in calibrated glass cylinders, while measurements of large quantities are made in metal or wooden tanks. In most cases, the containers are right cylinders. The volume is computed as the area of the base times the altitude to the point of measurement.

In containers of small cross section, the effect of surface tension is considerable and causes the formation of a *meniscus* at the top surface; in the simple case of water in glass, the water is drawn up at the edges, where it meets the glass. To avoid inaccuracy, measurement is made to the level of the center of the meniscus, Fig. 22. Mercury exhibits a negative effect, with the liquid depressed at the edges; the measuring line is tangent to the top of the meniscus.

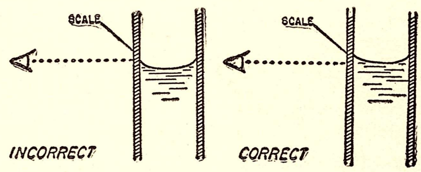
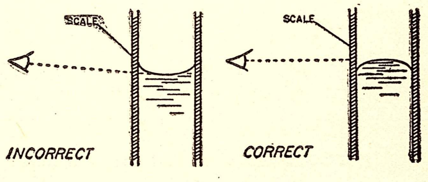


Fig. 22

In making observations, the observer must take care that his line of sight be perpendicular to the cylinder in which the liquid is contained. Since the scale divisions are on the outside of a measure, while the point in the liquid to which the measure is taken is at the center or axis of the cylinder, an incorrect angle of observation can cause a faulty reading. This source of error is known as *parallax*. Parallax may enter as a source of error



in all types of measurements where the point of measurement is not coincident with the scale.

Pipettes are flasks for volumetric measurement. The type illustrated in Fig. 23 is filled, eye-dropper fashion, the mouth being used to draw the liquid into the pipette until it is filled above the calibration mark. The finger is then placed on top to act as a valve, and the liquid meniscus level is allowed to drop to the calibration mark. The pipette will then *deliver* the volume for which it was graduated, despite the fact that a small part of the original contents will adhere to the inside of the pipette. Some types of graduated flasks used for mixing purposes are graduated to *hold* the volume indicated, but will *deliver* a lesser volume. This distinction must be observed carefully when precise measurements are being made.

Fig. 23

Equations of units

When one is asked to find the number of inches contained in seven feet, the process of calculation is quite simple: $7\times12=84''$; on the other hand, when one is asked to find the number of cubic inches in a cubic yard, he frequently becomes quite entangled in arithmetic, yet the principle underlying both calculations is the same, and a thorough understanding of it will enable one to make with ease what may seem to be quite involved conversions. We shall now investigate that principle.

The numbers, 7 and 12, employed in the calculation of the number of inches in seven feet have definite units for which they stand; 7 stands for feet (ft.) and 12 stands for inches per foot $\left(\frac{\text{in.}}{\text{ft.}}\right)$, *per* meaning "divided by". Thus, when multiplying $7\times12=84''$, we are actually combining units in the form of $\cancel{\text{ft.}}\times\frac{\text{in.}}{\cancel{\text{ft.}}}=\text{in.}$ An equation written in this form, containing units alone, is known as an equation of units.

Now, notice that, if we applied the rules for the cancellation of equal factors in the numerator and denominator during the multiplication of fractions to the cancellation of units in the numerator and denominator of the equation of units above, we should have: $\text{ft.}\times\frac{\text{in.}}{\text{ft.}}=\cancel{\text{ft.}}\times\frac{\text{in.}}{\cancel{\text{ft.}}}=\text{in.}$, which, we know, are the units of the answer. *This operation of cancelling units is always valid*, and it is because of this universal validity that equations of units are so valuable.

The rules of algebra which apply to manipulations of numbers and characters on both sides of an equation are not valid for the manipulation of units and numbers in the statement of an equivalence.

Unfortunately, an = symbol is misused to denote equivalence; *i.e.*, 7 ft. = 84 in. By applying algebraic manipulation to get the units on one side of the equation and numbers on the other, we obtain $\frac{\text{in.}}{\text{ft.}}=\frac{1}{12}$. This result obviously is inverted; the ratio $\frac{\text{in.}}{\text{ft.}}$ is 12. This inconsistency is circumvented by noting that, when the smaller unit is in the numerator, the ratio of units is greater than one; and when the smaller unit is in the denominator, the ratio is less than one. Some mathematicians carefully note the distinction between algebraic equality and equivalence of units by using the symbol, =, only when the numerical values on both sides of the symbol are the same and by using the symbol, $\approx$, to indicate "equivalent to".

To take another elementary example, suppose we wished to convert the units of a volume which we had measured in cubic centimeters to units of liters. We should say: cc×X = liters, where X represents the units of the unknown conversion factor by which we must multiply cc to get liters. Obviously

$$X=\frac{\text{liters}}{\text{cc}} \text{ or } \frac{1}{X}=\frac{\text{cc}}{\text{liters}}.$$

Now, we know that there are 1000 $\frac{\text{cc}}{\text{liter}}$; hence, $\frac{1}{X}=1000$, or $X=\frac{1}{1000}$. Hence, $\text{cc}\times\frac{1}{1000}=\text{liters}$.

We might have ascertained X more simply if we had asked ourselves this question: "By what must we multiply cc so that the cc units will cancel out and liters will remain in the numerator?" Obviously,

$$\text{cc}\times\frac{\text{liters}}{\text{cc}}=\cancel{\text{cc}}\times\frac{\text{liters}}{\cancel{\text{cc}}}=\text{liters, or}$$

the units of our conversion factor must be $\frac{\text{liters}}{\text{cc}}$ as we have already discovered by less direct reasoning.

We can also apply the device of breaking up an unknown ratio of units into two or more known ratios. By making numerical substitutions into these known ratios, we can solve for the desired one.

By learning this method thoroughly, one may evaluate for himself many conversion factors and thus become largely independent of conversion tables.

Let us take another and more involved example. Suppose we are refueling a bomber through a pump which delivers 150 cubic inches of gasoline per second to the fuel tanks. If the tanks have a capacity of 2000 gal. and are empty at the start, how many hours will it take to fill them?

By the preceding argument,

$$\frac{\text{in.}^3}{\text{sec.}}\times\frac{\text{sec.}}{\text{hr.}}\times\frac{\text{gal.}}{\text{in.}^3}=\frac{\cancel{\text{in.}^3}}{\cancel{\text{sec.}}}\times\frac{\cancel{\text{sec.}}}{\text{hr.}}\times\frac{\text{gal.}}{\cancel{\text{in.}^3}}=\frac{\text{gal.}}{\text{hr.}}$$

It is often convenient in such extended calculations to write the numerator of the entire expression above one long line, and the denominator below, as follows:

$$\frac{\text{in.}^3\times\text{sec.}\times\text{gal.}}{\text{sec.}\times\text{hr.}\ \times\text{in.}^3}=\frac{\text{gal.}}{\text{hr.}} \qquad \text{I}$$

This expression gives us an answer in units of $\frac{\text{gallons}}{\text{hour}}$, and we wish our answer to be in units of hours. Hence, hours = total gallons to be delivered $\times\frac{\text{hr.}}{\text{gal.}}=\frac{\text{total gallons to be delivered}}{\frac{\text{gal.}}{\text{hr.}}}$. By substitution of equation I,

$$\frac{\text{total gal.}}{\frac{\text{in.}^3}{\text{sec.}}\times\frac{\text{sec.}}{\text{hr.}}\times\frac{\text{gal.}}{\text{in.}^3}}=\frac{\text{total gal.}\times\text{sec.}\times\ \text{hr.}\times\text{in.}^3}{1\times\text{in.}^3\times\text{sec.}\times\text{gal.}}=\text{hr.} \qquad \text{II}$$

We may now insert the given values:
Total gallons = 2000;

$$\frac{\text{sec.}}{\text{in.}^3}=\frac{1}{\frac{\text{in.}^3}{\text{sec.}}}=\frac{1}{150};\ \frac{\text{hr.}}{\text{sec.}}=\frac{1}{\frac{\text{sec.}}{\text{hr.}}}=\frac{1}{3600};\ \frac{\text{in.}^3}{\text{gal.}}=231.$$

Therefore, by substitution in equation II,

$$\text{hr.}=\frac{2000\times\ 1\ \times\ 1\ \times231}{1\ \times150\times3600\times\ 1}=0.855\text{ hrs.}$$

By dint of some practice, one may learn to handle such calculations very rapidly and may, at any time, by referring to the dimensional equation, be certain that he has not multiplied by a factor when he should have divided by it, and *vice versa*.

TEST YOUR ABILITY TO CONVERT UNITS

4 There are 2.54 centimeters in one inch. How many inches are there in one meter? $\left(\frac{\text{cm.}}{\text{meter}}\times\frac{\text{in.}}{\text{cm.}}=\frac{\text{in.}}{\text{meter}}\right)$

5 A body is moving with a speed of 90 cm. per second. Express this speed in miles per hour. $\left(\frac{\text{cm.}}{\text{sec.}}\times\frac{\text{in.}}{\text{cm.}}\times\frac{\text{ft.}}{\text{in.}}\times\frac{\text{mi.}}{\text{ft.}}\times\frac{\text{sec.}}{\text{min.}}\times\frac{\text{min.}}{\text{hr.}}=\frac{\text{mi.}}{\text{hr.}}\right)$

6 The acceleration of gravity for engineering purposes is taken as $32.2\ \frac{\text{ft.}}{\text{sec.}^2}$. Express this in $\frac{\text{cm.}}{\text{sec.}^2}$. $\left(\frac{\text{ft.}}{\text{sec.}^2}\times\frac{\text{in.}}{\text{ft.}}\times\frac{\text{cm.}}{\text{in.}}=\frac{\text{cm.}}{\text{sec.}^2}\right)$

7 The vertical distance covered by a freely falling body is related to the duration of the fall through the expression, $s=\frac{1}{2}gt^2$, where g is the acceleration of gravity, t is the time elapsed since the fall began, and s is the distance covered in the time, t. Air friction is neglected. A parachute jumper waits about three seconds after leaving an airplane before pulling the parachute rip-cord. (a) How far does the jumper fall in three seconds? Use the value of g as $32.2\ \frac{\text{ft.}}{\text{sec.}^2}$. (b) How far would he have fallen in two seconds? (c) How far would he have fallen if he had waited but one second?

FUNDAMENTAL DIMENSIONS AND DERIVED QUANTITIES

In the statements and equations of the laws of mechanics, and hence in mechanical measurements, only three *fundamental quantities* exist. They are *length* (L), *time* (T), and *mass* (M). They are fundamental because the units in which they are measured must be fixed arbitrarily. All other quantities which occur in mechanical calculations, such as force and velocity, may be expressed in terms of M, L, and T.

Length and time have been described above. Mass is not so familiar and requires some explanation. We think, ordinarily, that, when we weigh a body, we are discovering how much matter there is in the body. Generally speaking, this is not so, for, if we were to take a heavy object, suspend it from a spring scale and carry it aloft in a high-altitude airplane, we should discover that, the higher we rose, the lighter the object would become. Furthermore, we should notice, upon returning to the earth, that the weight of the object returned to its original value. To the best of our knowledge, the amount of material in the object did not change during the flight, and yet it was observed that the weight of the object did. We conclude, therefore, that the weight of a body is not a measure of the amount of material in it. (Actually, the weight depends on the gravitational pull of the earth.) To obtain an adequate unit to represent the amount of material, we must discover some property of the object which depends on its quantity, but not on its location. This is the mass of the object, and its relation to *weight* is illustrated as follows:

If, in the above illustration of the high-altitude plane, we had released the object near the roof of the cabin and measured the acceleration* with which that object fell to the floor, we should have noticed that this acceleration was less than the acceleration in a similar fall nearer the earth, yet the ratio of the acceleration in free fall of the object at some high altitude to the acceleration in free fall of the object near the earth's surface would be found to equal the corresponding ratio of the weights of the object measured with a spring scale in the same two places.

That is,

$$\frac{\text{weight in air}}{\text{weight on earth}}=\frac{g \text{ in air}}{g \text{ on earth}};$$

---

* Acceleration is the rate of change of velocity with time, and is signified by the letter, a. It has units in the English system of feet per second per second $\left(\frac{\text{ft.}}{\text{sec.}^2}\right)$. The average acceleration experienced by a body at the beginning of its fall near the earth is $32.2\left(\frac{\text{ft.}}{\text{sec.}^2}\right)$. This universal acceleration due to the earth's gravitational attraction is a constant for any one place and varies only slightly over the surface of the earth. It is less at points above and below the earth's surface than it is just at the surface. It is signified by the letter, g, instead of the general abbreviation, a. Galileo (1564–1642) demonstrated that two balls of much different weight dropped together from the leaning tower of Pisa reached the ground at the same time. The velocity of each had changed from zero at the beginning to the same definite velocity at the end; *i.e.*, they were both subject to the same acceleration. In our discussion, we must limit ourselves to short observations, where the effect of air resistance is negligible. A penny and a feather dropped in air have the same acceleration at the very start, in any case; but this acceleration is constant throughout the fall only when we make the observation in a vacuum, where the body's resistance is wholly that of inertia, and no air resistance is present.

by transposition,

$$\frac{\text{weight in air}}{g \text{ in air}} = \frac{\text{weight on earth}}{g \text{ on earth}} = \text{a constant called } \textit{mass}.$$

This relationship is written $M = \frac{W}{g}$. In its general form, $F = Ma$, it expresses Newton's second law of motion.

Units of force and mass

An awkward situation exists because there are many systems of force and mass, and few are completely in order. The systems in common use will be presented here. This table should be read, "A force of one pound will accelerate a mass of one slug one foot per second per second."

	FORCE	MASS	a
British Engineering	1 pound	1 slug	$1 \frac{\text{ft.}}{\text{sec.}^2}$
British absolute	1 poundal	1 pound mass	$1 \frac{\text{ft.}}{\text{sec.}^2}$
C.G.S.	1 dyne	1 gram	$1 \frac{\text{cm.}}{\text{sec.}^2}$
M.K.S.	1 newton	1 kilogram	$1 \frac{\text{meter}}{\text{sec.}^2}$

From our fundamental quantities (M, L, T), we can derive the dimensions of the other mechanical quantities, which are simply combinations of the fundamental quantities. An area is a length times a width; since both these factors have the dimensions of L, the dimensions of the area are:

$$L \times L = L^2.$$

Velocity is distance per unit time:

$$V = L \times \frac{1}{T} = \frac{L}{T} = LT^{-1}$$

Acceleration is change of velocity per unit time:

$$a = V \times \frac{1}{T} = \frac{L}{T} \times \frac{1}{T} = \frac{L}{T^2} = LT^{-2}.$$

Force is equal to mass times acceleration:

$$F = Ma = MLT^{-2}.$$

Work and energy are equal to force times distance:

$$W = F \times L = MLT^{-2} \times L = ML^2T^{-2}.$$

Power is work per unit time:

$$P = W \times \frac{1}{T} = ML^2T^{-2} \times \frac{1}{T} = ML^2T^{-3}$$

TEST YOUR ABILITY TO MEASURE VELOCITY

8 The velocity of a large projectile is measured by the time required for the shell to pass between two screens which are placed in its path a known distance apart. The time required for a certain projectile to pass between two screens 200 ft. apart is 0.0690 seconds. What is the *average* velocity of the shell in its flight over this distance? $\left(V = \frac{L}{T}\right)$

Measuring instruments for forces

Since the most commonly measured forces are weights, we shall here discuss instruments for weighing. These, with slight modifications, are used to measure forces other than gravitational.

In the *beam balance*, the unknown weight of an object is determined by adding up the quantity of standard weights required to establish a neutral condition. Very sensitive chemical beam balances are made for small weights up to about 100 grams. Large beam balances for heavy weights are no longer the fashion because they are inconvenient.

The *steelyard* (Fig. 24), often used to measure sides of meat, balances the unknown weight when the sliding weight is in proper position along the calibrated arm. Here the weight being measured and the sliding weight are very unequal, but a condition of balance can be established because of the greater leverage of the sliding weight. When the scale is in balance, the weight of the object is read directly on the calibrated arm.

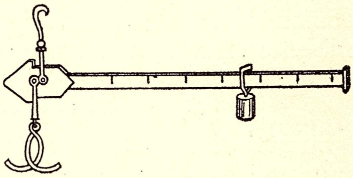


Fig. 24

The *Roberval* type of *pan balance* (see page 44) is seen in use by many druggists and is very useful for weighing small parts rapidly. It has two distinct advantages over the simple balance:

a The pans do not swing but move only vertically;

b The weights need not be centered on the pans.

In some types, a calibrated arm may supplement the weight pan, or replace it entirely. A rugged design incorporating these advantages is the familiar *platform scale*, used to measure barrel-sized bulks of from 50 to 1000 pounds weight.

Spring scales are so popular and familiar that we need not discuss them in detail here.

TORQUE

Torque (often called *moment*) is the twisting or rotating tendency of a force applied to an object. Torque is computed by the mathematical product of the applied force and the shortest (perpendicular) distance from the line of action of the force to the point about which the torque is to be measured. It is different for various points on the object and is zero for all points exactly on the line of action of the force.

As energy transferred by rotary motion is equal to torque times angular displacement (in radian measure), a complete understanding of torque is necessary for the computation of rotary energy and power.

Frequently, when we are concerned with torques, the objects to which the forces are applied are supported by an axle and bearing. We are then concerned with the torque which is produced by a force about the point of suspension.

To evaluate a torque, we have but to measure a force and a distance and multiply them together. Methods for making these two measurements have already been described. The dimensions of torque are force times distance (FL) or $\left(\frac{ML^2}{T^2}\right)$, commonly expressed in units of lb.-ft., ounce-in., dyne-cm., or newton-meters. This convention of putting the force unit before the distance unit is adhered to by many engineers to avoid confusion with energy units which are also force times distance; *i. e.*, lb.-ft. is used for torque; ft.-lb., for energy. Practical methods for measuring the torques of spinning shafts are described under "Power", page 566.

DENSITY AND SPECIFIC GRAVITY

Scientists use the one word, *density*, to define *mass* per unit volume. Some engineers and practical men use density as *weight* per unit volume. Here is another stumbling-block for students. The two methods can easily be distinguished by stating the units, or using the full names, *mass density*, or *weight density*.

Specific gravity is the ratio of the weight of a body to the weight of an equal volume of water in consistent units. A little reflection will show that this ratio is the same as the ratio of mass density of the body to mass density of water. Specific gravity has no units in the proper sense; it is a purely numerical ratio. Many arbitrary scales are used by different industries, and, since the numbers are generally used for comparisons rather than for complex computations, the use of irregular units does no harm if the limitations are understood.

Densities and specific gravities of solids and liquids may be determined by weighing a known volume of the substance, or by an application of Archimedes' principle. This principle states, "An immersed body is buoyed up by a force equal to the weight of the displaced liquid".

The hydrometer, usually blown from glass, has a weighted bottom and a long slender neck as sketched in Fig. 25. When placed in a liquid, the hydrometer will float at a position where it displaces a volume of fluid having a weight equal to its own. The lighter the fluid, the farther the hydrometer will sink. The density or specific gravity is read directly at the liquid surface on a calibrated scale

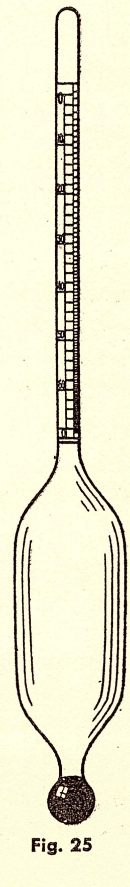


Fig. 25

inside the slender glass neck. The temperature of the liquid must be the same as that for which the hydrometer was calibrated.

The *Westphal balance* is used for precise measurements of specific gravity of liquids. Here a beam-plummet-counterweight combination is balanced in air initially. The plummet, suspended on a fine wire, is then completely submerged in the fluid of unknown density. The more dense the liquid, the more the beam will be thrown out of balance because of the buoyant force. Small riders are moved along the graduated beam until balance is again obtained, and the instrument is so calibrated that the desired specific gravity is then read directly from the position of the riders. As the riders supplied have weights in the ratio of one to ten, the answer is read decimally in the order of decreasing weight of rider.

TEST YOUR ABILITY TO MEASURE DENSITY AND SPECIFIC GRAVITY

9 A piece of iron which weighs 100 lbs. in air weighs 85.9 lbs. when it is immersed in water. What is the specific gravity of the iron? (Weight of unknown divided by weight of an equal volume of water.)

10 Knowing that water weighs 62.4 lbs. per cubic foot, find (a) the volume of the iron, and (b) the specific weight of the iron in $\frac{\text{lb.}}{\text{ft.}^3}$.

Pressure is the force acting over a unit area of surface. Typical units are $\frac{\text{pounds}}{\text{square in.}}$, $\frac{\text{dynes}}{\text{square cm.}}$. Pressure may be due to the action of a solid or a fluid. In a closed hydraulic system a small force acting on a small area of fluid causes a relatively high pressure. This is utilized in fluid-drive gun turrets, where the larger piston attached to the gun will experience a much greater force on the gun than the smaller activating force of the pump.

TEST YOUR ABILITY TO MEASURE PRESSURE

11 The specific gravity of mercury is 13.6. What will be the atmospheric pressure in $\frac{\text{lb.}}{\text{in.}^2}$ on a day when it supports a column of mercury 29.00″ high?

$$\frac{\text{lb.}}{\text{ft.}^3} \times \frac{\text{in.}}{\frac{\text{in.}}{\text{ft.}}} \times \frac{1}{\frac{\text{in.}^2}{\text{ft.}^2}} = \frac{\text{lb.}}{\text{in.}^2}$$

A barometer is made from a glass tube, closed at one end and filled with pure mercury. With special care, the small amount of air which clings to the inside surface of the tube is expelled. The open end is closed temporarily while the tube is turned up into a mercury reservoir.

When the submerged end of the glass tube is again opened, the column will fall until it balances the pressure on the mercury surface. This is shown in Fig. 26. We say that the atmospheric pressure "supports a column of mercury" so many units high (29.9 inches or 760 m.m. at normal conditions of temperature and weather and at sea level). Normal atmospheric pressure is about $14.7 \frac{\text{lbs.}}{\text{in.}^2}$.

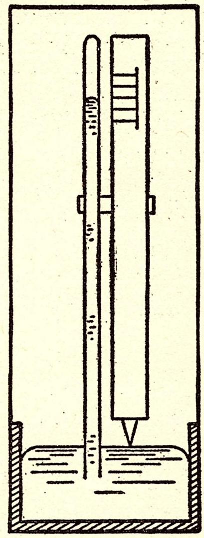


Fig. 26

Atmospheric or barometric pressure is caused by the gravitational attraction of the earth on all the air molecules above the point of measurement. These tiny particles exert a continual pressure on all surrounding objects. Their pressure is greatest at low altitudes, just as water pressure increases with depth.

In any meter using the height of a column of liquid to measure pressure, the rule which applies is: the difference in pressure between one end of the column and the other is equal to the specific weight of the liquid used in the tube times the difference in column heights (marked H in Figs. 28 and 29). Consistent units must be used throughout this operation, *i. e.*, if the column height is expressed in *inches*, the specific weight must be in units of weight per square *inch*.

Consider a mercury barometer with a height of 30 inches. The specific gravity of mercury is 13.6; the specific weight of water is $62.4 \frac{\text{pounds}}{\text{ft.}^3}$; hence, the specific weight of mercury is $13.6 \times 62.4 \frac{\text{pounds}}{\text{ft.}^3}$. The difference in pressure, top of mercury column to bottom, is $13.6 \times 62.4 \frac{\text{pounds}}{\text{ft.}^3} \times \frac{30}{12} \text{ ft.} = 2122 \frac{\text{pounds}}{\text{ft.}^2} = 14.74 \frac{\text{pounds}}{\text{in.}^2}$.

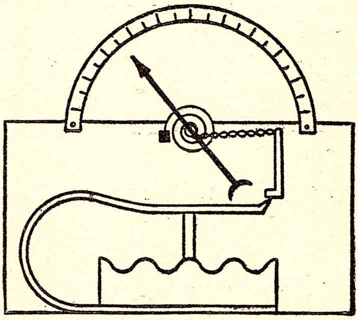


Fig. 27

The pressure at the top of the mercury column in a barometer, and also a *closed-tube manometer* (Fig. 28), is a high vacuum; *i. e.*, zero absolute pressure. The pressure difference between the two ends is called *absolute pressure* since the datum is absolute zero pressure.

The *aneroid barometer* (Fig. 27) is a portable instrument which indicates changes in atmospheric pressure by the deflection of one or more partially-evacuated, metallic bellows.

The *open-tube manometer* (Fig. 29), is convenient to measure pressures relative to atmospheric pressure. The pressure difference between the two ends of the column when one end is open to the atmosphere is called *gage pressure*. An open-tube manometer containing a liquid of known specific weight is a primary standard of gage pressure.

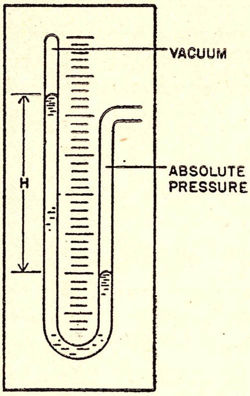


Fig. 28

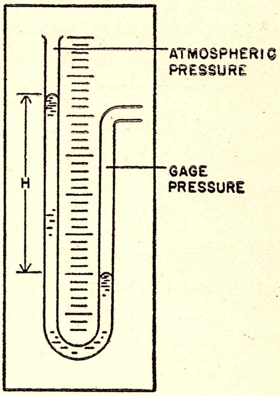


Fig. 29

A *dead weight tester* (Fig. 30) is also a primary standard of gage pressure. It consists of a piston-weight-cylinder combination. The working fluid is oil, and the pressure is computed by dividing the weights applied by the area of the piston. This form of tester is used almost exclusively to calibrate engine indicators and Bourdon tube gages, described below.

A *Bourdon tube gage* (Fig. 31) consists of an elastic tube (usually steel or phosphor bronze) of oval cross section. The tube is bent in a circular arc. One end is coupled to the unknown pressure; the other end is closed. The closed end is attached to a magnifying linkage

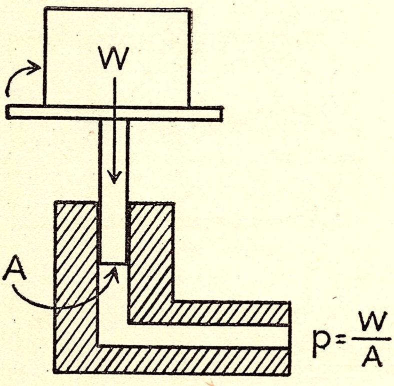


Fig. 30

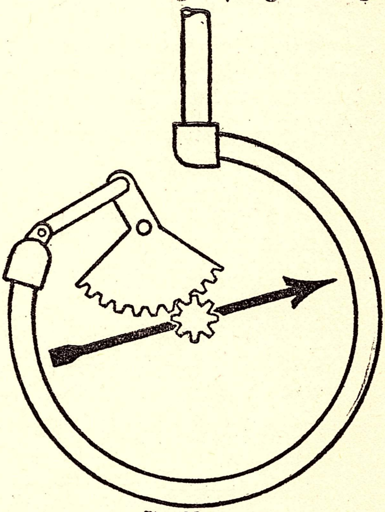


Fig. 31

which actuates a pointer. As pressure is applied inside the tube, the cross section approaches a circle. The tube tends to straighten, moving the closed end and hence the pointer.

TEST YOUR ABILITY TO MEASURE PRESSURE BY MANOMETER

12 An open-tube manometer filled with water is used to measure the pressure at a certain point on an airplane model during a wind-tunnel test. The difference between the water levels (H) in the two sides is found to be 8.71 inches (see Fig. 29), the level in the open side being the higher. (a) What is the pressure in $\frac{\text{lbs.}}{\text{ft.}^2}$ indicated by the gage? $\left(\text{Water weighs } 62.4\ \frac{\text{lbs.}}{\text{ft.}^3}\right)$. (b) If the atmospheric pressure on the day of the test is $14.686\ \frac{\text{lb.}}{\text{in.}^2}$, what is the absolute pressure at the point on the airplane model to which the manometer is connected? (c) What would be indicated if the level of the water in the open side of the manometer were lower than the level of the water in the side to which the tube is connected?

Measuring instruments of stress and strain in materials

A steel control rod of an airplane which performs its function through the motion of its own actuating lever will be satisfactory if it has sufficient tensile strength to withstand the load. The structural members which support mechanical bearings, however, must not only have strength to withstand the load, but must also possess sufficient rigidity to prevent misalignment and binding of moving parts. The units which define such qualities as strength and rigidity and the methods of their determination are discussed below.

Elasticity is a general term used to describe the tendency of a deformed body to resume its initial shape after the deforming forces have been removed.

Stress is the internal force acting over a unit cross section of material, is associated with an external force on the material, and may be either a tensile, a compressive, or a shearing force. In simple cases, such as the control rod mentioned above, the total force is the same at each end and is distributed uniformly over the cross section. We say in this case that the stress is constant over the cross section and along the length of the rod.

When forces on a rod act in such a way as to *bend* it, the stress distribution becomes complex; it involves compressive stress on one side of the rod, and tensile stress on the other side. A shear stress appears, which is maximum in the middle.

$$\text{Stress} = \frac{\text{force}}{\text{area over which it is distributed uniformly}}.$$

It generally is expressed in units of pounds per square inch $\left(\frac{\text{lb.}}{\text{in.}^2}\right)$.

Tensile strength is the stress at which a sample will fail in simple tension.

Strain accompanies a stress. It is the deformation of a unit length of the material.

$$\text{Strain} = \frac{\text{change in length}}{\text{length considered}}.$$

It is a dimensionless ratio, since the original length and the change in length are measured in the same units. In the middle of the seventeenth century, Robert Hooke discovered that a piece of metal deformed an amount directly proportional to the force applied, as long as the metal was not deformed to the point where it took a permanent set. In terms of stress and strain, *Hooke's Law* is:

$$\text{Stress} = \text{strain} \times \text{constant}.$$

The constant will have units of force per unit area and is different for different metals and alloys. It is named after the English physicist, Thomas Young.

$$\text{Young's Modulus, } E = \frac{\text{stress}}{\text{strain}}.$$

Experimental data which are the bases for computation of the quantities above are obtained from *testing machines*. Testing machines are usually power-driven and are essentially hydraulic or screw presses, modified to apply either tensile or compressive forces to a specimen. The forces applied in hydraulic machines are read directly from a calibrated gage of the Bourdon type. Calibrated balancing arms, similar to the type used in platform scales, are used to indicate the forces applied to the specimen in the screw-driven testing machine.

Deformations are usually measured by dial gages attached to the specimen. Often a stress-strain diagram is plotted for the test specimen.

Work, energy, and power

The *work* done in moving an object is the mathematical product of the force applied to the object during motion times the distance moved by the object. Work is done only when something is moved. (This physical idea is perhaps contrary to the popular notion of work, for a man could tug and pull at a large rock all day yet, if he did not move the rock, a physicist would say that he had done no work. The man, however, might exhaust himself in the process. This is purely a matter of definition.)

If the force applied to move an object remains *constant* during motion, the work done is simply the constant force times the distance moved. If, on the other hand, the force causing the motion *varies* during the motion (as when one has to push a heavy car harder to start it than he has to push to keep it moving), the work done is equal to an *average* force times the distance moved.

Similarly, when the application of a torque causes an object to rotate, work is being done by one body on another. In this case, the

work done is the mathematical product of an average torque applied during rotation times the angle (expressed in radians) through which the object rotates.

ENERGY

Energy is the ability to do work. One of the fundamental laws of physics is that of the "Conservation of Energy". This law states that energy can be neither created nor destroyed. Energy can, however, be transferred from one body to another, or from one form to another. For example, all the work done in accelerating a body from rest to some speed will be done again by the body on whatever medium slows it down to rest, provided that the elevation of the body is unaltered. A body may possess energy in any of the several forms.

Potential energy is possessed by a body when it is in a position where it may be permitted to descend for some distance under the influence of gravity. If the body were to fall, it would descend with the force of gravity (the body's weight) acting on it and would acquire energy of motion equal to the distance fallen times its weight.

Kinetic energy is energy of motion. The kinetic energy of a body moving in a linear path is found by the formula, k. e. $=\frac{1}{2}MV^2$, where M is the mass of the body, and V is the velocity of the body in corresponding units.

The kinetic energy acquired by rotating bodies, such as turbines and flywheels, is given by the expression, k. e. $=\frac{1}{2}I\omega^2$, where I is the second moment of mass (moment of inertia) of the body, and ω is the angular velocity (radians per second) of rotation.

Energy may be imparted by forces other than gravitational forces. The majority of engineering problems deal with energies of this sort, since most machinery operates on, or transmits, mechanical energy which has been converted from electrical or thermodynamical forms rather than from gravitational forms.

Let us examine the dimensions of energy. We have, in the preceding paragraphs, set energy equal to the following:

$$\text{Energy} = \text{force}\left(\frac{ML}{T^2}\right)\times \text{distance } (L) = \frac{ML^2}{T^2};$$

$$\text{Energy} = \text{torque}\left(\frac{ML^2}{T^2}\right)\times \text{angle (no dimensions)} = \frac{ML^2}{T^2};$$

$$\text{Energy} = \frac{1}{2}MV^2 = \frac{ML^2}{T^2};$$

$$\text{Energy} = \frac{1}{2}I\omega^2 = (ML^2)\times\left(\frac{1}{T}\right)^2 = \frac{ML^2}{T^2}.$$

Thus, we see that energy has the dimensions of $\frac{ML^2}{T^2}$. This is usually expressed as force units times distance units, such as ft.-lbs., in.-lbs., newton-meters (given the special name of "Joules"), and dyne-cm (given the special name of "ergs").

Before proceeding to a discussion of methods of measuring energy, we shall do well to grasp the concepts of efficiency and mechanical advantage. *Efficiency*, as it applies in mechanics, is defined as the ratio of the energy output of a machine or device to the corresponding energy input; that is, $\text{efficiency}=\left(\frac{\text{energy out}}{\text{energy in}}\right)$. Efficiency has no units; it is a dimensionless ratio. By the law of conservation of energy, we know that the difference between the energy input and the corresponding energy output represents energy which has been diverted, during transfer through the machine, into frictional heat. It is generally expressed as a percentage. An 80% efficient machine is one which delivers, in the desired form, 80% of the energy supplied it. The remaining 20% of input energy, usually wasted in frictional losses, is accounted for in heat. No machine is ever 100% efficient. Steam engines are seldom better than 20% efficient. Gasoline engines are only slightly better.

In practical measurements of the energy output of machines, it is often simpler to measure input energy and make corrections for efficiency than to measure the output directly.

Illustrative Example

A reservoir contains two million cubic feet (2×10^6 ft.^{3}) of water. Water weighs 62.4 pounds per cubic foot. The water is to flow through a pipe a vertical distance of 100 feet and there be employed to drive a water wheel. Assuming the piping system to be 85% efficient (15% of the potential energy of the water is required to overcome friction in the pipe) and the water wheel to be 80% efficient (20% of the energy delivered to the wheel is wasted in turbulence and in friction in the mountings), how much energy will have been delivered by the wheel when all the water in the reservoir has flowed through the system?

The potential energy of the water relative to the wheel is equal to the weight of the water times the distance vertically to the wheel, or potential energy $=(2\times10^6\times62.4\times100)$ ft.-lb. By the law of conservation of energy, this energy, less friction losses, will be delivered to the wheel in the form of kinetic energy. With 15% losses, the energy delivered to the wheel will be potential energy$\times$efficiency of piping system $=(2\times10^6\times62.4\times100\times0.85)$ ft.-lb. Of this amount, 20% will be lost in the bearings of the wheel. Hence, the wheel will deliver $(2\times10^6\times62.4\times100\times0.85)\times0.80=85\times10^8$ **ft.-lb.**

$$\text{ft.}^3\times\frac{\text{lb.}}{\text{ft.}^3}\times\text{ft.}\times\frac{\text{ft.-lb.}}{\text{ft.-lb.}}\times\frac{\text{ft.-lb.}}{\text{ft.-lb.}}=\text{ft.-lb.}$$

Sometimes it is possible to make energy measurements directly. An "indicator card", Fig. 32, (the name is taken from the steam-engine indicator, which draws the figure) is useful in measuring the energy per cycle acquired by the piston of a reciprocating steam engine. It is a graph of the steam pressure in the cylinder versus the distance moved by the piston. The pressure at any instant times the piston area is the force acting on the piston at that instant; the average force on the piston (average with respect to distance) times the distance moved by the piston during the *power* stroke is the work done by the steam. Notice in the diagram that the piston does work *on* the steam during the *exhaust* stroke; the piston takes energy from the flywheel and pushes the exhaust steam from the cylinder. During the power stroke, the force on the piston is in the *same* direction as the motion of the piston; and during the exhaust stroke, the force on the piston by the steam is in opposition to the direction of the piston. The area of the pressure-distance loop $\left(\frac{\text{force}}{\text{area}}-\text{distance loop}\right)$ is a measure of the energy of one complete cycle (power and exhaust stroke). The following computation is necessary:

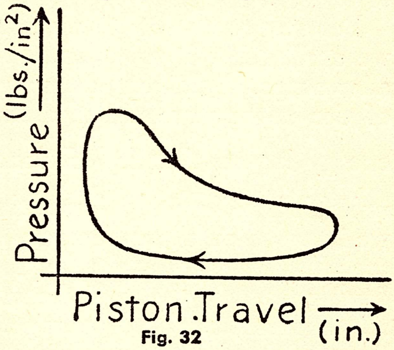


Fig. 32

$$\text{piston area}\times\frac{\text{pressure}}{\text{inch of ordinate}}\times\frac{\text{piston travel}}{\text{inch of abscissa}}\times\text{area of graph}=\begin{matrix}\text{energy}\\ \text{per cycle}\end{matrix}$$

$$\text{in.}^2\times\frac{\frac{\text{lb.}}{\text{in.}^2}}{\text{in.}}\times\frac{\text{in.}}{\text{in.}}\times\text{in.}^2=\text{in.}\times\text{lb.}$$

The area of the indicator-card loop (in square inches) times the product of the scales of the indicator-card axes (in pressure and travel units per inch of axis length) times the piston area gives the energy per cycle imparted to the piston by the steam. The area of the loop is read most easily by the Amsler Polar Planimeter. The *Coffin averager* may be employed to find the average pressure during the cycle (mean effective pressure, abbr. m.e.p.). From this and the piston travel and area, the energy per cycle may be found by a simpler computation.

Energy per stroke = (m.e.p.) × piston area × piston travel.

The idea of the conservation of energy may be found useful in problems which seem at first to be unrelated to energy. For example,

suppose we wish to measure the *velocity* of a bullet at some point in its trajectory. We can make direct velocity measurements in several ways which are quite difficult, as they all will require the accurate measure of a very short interval of time. A simple method is to measure the *energy* of the projectile and then to calculate the velocity through the expression, k.e. $=\frac{1}{2}MV^2$. The method is simple because it depends essentially on measurements of weights and distances only. The "ballistic pendulum" (Fig. 33) is the measuring instrument used. The bullet is fired into the block, shown as B in the accompanying sketch. The bullet imbeds itself in the block, imparting kinetic energy and causing the pendulum to swing. The energy imparted to the block by the bullet is entirely kinetic but, as the block rises in its swing to its point of maximum deflection, B', work is done in raising the block through the distance, h. This work, and the energy required to perform it, must equal $h \times$ the weight of the block. Since B' is the point of maximum deflection, the velocity there must be zero; hence, the block has no more kinetic energy, its present energy being entirely of the potential sort.

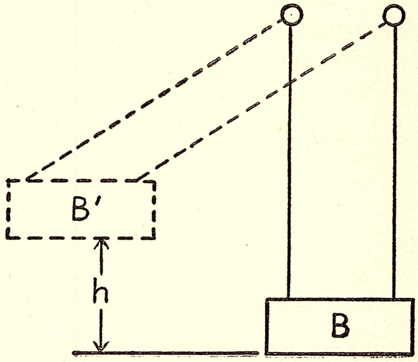


Fig. 33

By the law of conservation of energy, this amount of potential energy must equal the initial kinetic energy of the block at B, which, in turn, must equal the kinetic energy of the bullet just before striking the block. Therefore, we write the equation,

$$\frac{1}{2}MV^2 = h \times \text{weight of block},$$

or,

$$V = \left(\frac{2h \times \text{weight of block}}{M}\right)^{\frac{1}{2}}.$$

Note that, as the pendulum swings, although the total energy of the block remains constant, the form of the energy is oscillating continually from kinetic, at the bottom of the swing, to potential at the top of the swing. At intermediate positions, the block has both potential and kinetic energy. All oscillating systems exhibit a similar interchange of energy form.

Illustrative Example

Let us now work out a problem which involves the use of the ballistic pendulum to measure a bullet velocity. A .30-caliber machine gun fires a 172-grain bullet into the 400-pound block of a ballistic pendulum and raises the block 5.05 feet. The acceleration of gravity is $\frac{32.2 \text{ ft.}}{\text{sec.}^2}$.

There are 7000 grains to the pound. What was the velocity of the bullet just before striking the block? (Use equation for V derived on page 565.)

$$V=\left(\frac{2\times 5.05\times 400\times 7000\times 32.2}{172}\right)^{\frac{1}{2}}=2300\ \frac{\text{ft.}}{\text{sec.}}$$

$$\left(\frac{\text{ft.}\ \times\text{lb.}\times\text{grains}\times\ \text{ft.}}{\text{grains}\ \ \times\ \ \ \text{lb.}\ \times\text{sec.}^2}\right)^{\frac{1}{2}}=\left(\frac{\text{ft.}^2}{\text{sec.}^2}\right)^{\frac{1}{2}}=\frac{\text{ft.}}{\text{sec.}}$$

Hence, our units check, and the answer is $2300\ \frac{\text{ft.}}{\text{sec.}}$

TEST YOUR ABILITY TO MEASURE ENERGY

13 An indicator diagram was recorded on an air-draft engine, each piston of which had an area of 25 square inches. The measured area of the indicator diagram was 3.71 square inches. The pressure axis of the card was laid off to a scale of $400\frac{\text{lb.}}{\text{sq.in.}}$ per inch height of diagram; and the piston travel scale was 2.5 inches of stroke per inch of diagram. What was the energy per cycle developed in the cylinder?

$$\left[\text{in.}^2\times\frac{\text{in.}^2}{\text{cycle}}\times\frac{\frac{\text{lb.}}{\text{in.}^2}}{\text{in.}}\times\frac{\text{in.}}{\text{in.}}=\frac{\text{in. lb.}}{\text{cycle}}\right]$$

POWER

In the preceding paragraphs, we have discussed mechanical work as energy transfer. At no time did we specify how *rapidly* the transfer was taking place. Whether the time required for all the water in a reservoir to flow through the system is five hours or five days is of no consequence in a discussion of the work done. On the other hand, if we were to use such a system to drive a generator and produce electric power for public consumption, we should be concerned greatly with the rapidity of the transfer. Such problems bring up the subject of *power*, which may be defined as the rate of doing work. Power has the dimensions of energy transferred per unit of time, $\frac{ML^2}{T^2}\times\frac{1}{T}=\frac{ML^2}{T^3}$, and is expressed usually either in units of horsepower, or in watts. One horsepower is 33,000 ft.-lbs. per minute, or 746 watts. One watt, a metric unit, is one joule per second. Measurements of power are made much more frequently than measurements of energy. Energy measurements may be derived from power measurements by multiplying the power by the time during which that power is transmitted. The product is the amount of energy transferred in that time.

Energy of linear motion can be expressed in *certain* (constant force) cases as force × distance. *Power* in cases of linear motion is *always* equal to the force producing motion × the velocity of motion. *Energy* in rotational motion is expressible in cases of constant torque as

torque $\times$ angular displacement, measured in radians. *Power* in rotational motion is *always* equal to torque $\times$ angular velocity, $\frac{\text{radians}}{\text{sec.}}$. The case of rotary motion is by far the most common one in which power is measured, since most sources of power, such as turbines, steam and internal-combustion engines, and electrical machinery produce rotary motion. To compute the power developed by such machinery, we have but to measure the torque on the output shaft and the velocity of rotation of the shaft. These are converted to proper units and multiplied together.

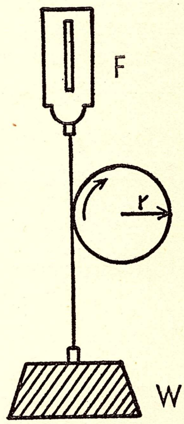


Fig. 34

The velocity of rotation of the shaft may be measured in several ways—by an electrical tachometer or by a stroboscope, for example. These give the rotational velocity in units of revolutions per second. This must be converted to units of radians per second by multiplying by 2π. $2\pi N = \frac{\text{radians}}{\text{sec.}}$, where N is in revolutions per second.

There are two types of *prony brakes* in common use (Figs. 34 and 35). Both consist essentially of a means of applying a measured frictional force at the periphery of a flywheel which is fastened to the rotating shaft. This frictional force times the radius in the flywheel (r) is the torque in the shaft. In Fig. 34, the difference between the spring balance reading, F, and the weight, W, is the frictional force applied at the rim of the wheel. This force times the radius of the flywheel is the torque. The torque is the mathematical product, $L \times F$. In this case, the brake beam must be counterbalanced so that the scales read zero when the flywheel is stationary, or the tare load must be considered.

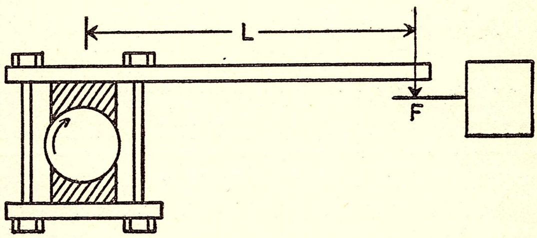


Fig. 35

To calculate the power in the first case, we write

$$\text{power} = \text{torque} \times \frac{\text{rad.}}{\text{sec.}} = (W-F) \cdot r \times 2\pi N \, \frac{\text{ft.-lb.}}{\text{min.}},$$

where W is the weight in pounds, F is the spring-balance reading in pounds, r is the radius in feet, and N is the number of revolutions per minute. To get the power in terms of horsepower (HP), we must divide by 33,000, since $33{,}000 \, \frac{\text{ft.-lb.}}{\text{min.}} = 1$ HP. Therefore, $\text{HP} = \frac{(W-F)\, r \times 2\pi N}{33{,}000}$.

In the second case, we write power $=2\pi NLF\ \frac{\text{ft.-lb.}}{\text{min.}}=\frac{2\pi NLF}{33{,}000}$ HP.

A method which often is employed to measure the output torque of aircraft engines consists of mounting the engine so that it is free to turn on an axis through the propeller shaft. The torque required to keep the engine from turning is measured at some known r.p.m., and the power is calculated as before.

TEST YOUR ABILITY TO MEASURE POWER

14 An interceptor airplane weighs 8000 lbs. Its rate of climb is 3,200 $\frac{\text{ft.}}{\text{min.}}$.

(a) How much power is required to lift the plane against gravity at this speed? $\dfrac{\text{lb.}\times\frac{\text{ft.}}{\text{min.}}}{\frac{\frac{\text{ft.-lb.}}{\text{min.}}}{\text{HP}}}=\text{HP}$ (b) If the engine is developing 1400 HP, where is the remaining power being consumed?

15 During a test of an aircraft engine, the torque on the engine was found to be 5250 ft.-lb., when the engine was running at 1900 r.p.m. (a) What horsepower was being developed? (b) Express this power in Kilowatts also.

16 A prony brake, similar to that shown in Fig. 35, is used to determine the output horsepower of an automobile engine. The length of arm, L, is 6 feet. The engine is turning over at 1100 r.p.m. with wide-open throttle; the reading of the scales is 48.5 lb. (a) What power (in horsepower) is being developed? (b) Express this in $\frac{\text{ft.-lb.}}{\text{sec.}}$.

TOLERANCES, SIGNIFICANT FIGURES, AND PRECISION OF MEASUREMENTS

A very common application of the theory of measurements may be seen in practice on almost every blueprint in the shop, where the precision required in the finished piece is either stated or implied. The length of a machine-gun firing pin may be designated as $2.7500'' \pm 0.0005$ or $\frac{2.7495''}{2.7505''}$. These are simply two methods of stating that the length of the finished pin is required to lie within 5 ten-thousandths of exactly two and three-quarters inches.

The measurement could not be given as $2\frac{3}{4}''$ or 2.75″ with no other understanding of what was expected, for several reasons. The part must be made with great care to fit in the assembly and function properly, and its finished length must be measured with an instrument capable of the precision required; *i.e.*, the part could not be measured with a wooden ruler. The above designation, $2.7500'' \pm 0.0005$, implies, however, that a part the length of which was 2.7504″ or 2.7496″ would function just as well as one made of length 2.7500″; a machinist would be wasting time in trying to be more precise. Further, the

part could not be made to a length of two and three-quarters inches *exactly*. One ten-thousandths of an inch is an extremely small length, and few shop micrometers are capable of dependable readings in this fourth decimal place. Moreover, two machinists measuring the same part will disagree frequently in the determination of this last digit. Even if one took all precautions to obtain a steel firing pin of *exactly* $2\frac{3}{4}''$ length, and examined the fruit of his labor under an electron microscope (a new device easily capable of sharp-image magnifications of 100,000 diameters), he would be greatly disappointed. A corner of the pin would be seen to move in worm-like fashion because of temperature transients unavoidable in ordinary room conditions.

The matter of shop tolerances is an elementary example of precision; it deals with a single measurement only. We shall now consider the more advanced theory involved in the precision of a result which depends on more than a single measurement, or on a single measurement taken to a power, or on a combination of sums and products. We shall see, for example, that, if a machinist caused an error of 5% in the diameter of a carburetor jet by using a slightly-bent drill, the error in the cross-sectional area (which depends on the square of the diameter) will be 10%.

For this study, we shall need to define some new terms:

Observer's Errors arise not only from negligence, but also from physical limitations of the observer. An observer given the task of determining the time required for an airplane to come to rest after landing would find it difficult to start and stop the watch at the proper instant without considerable error due to his judgment and reaction time.

Instrument Errors arise from faulty equipment, such as a steel tape which may have been run over by an automobile and stretched beyond its elastic limit. Other instances are: surveyor's wooden rods, which change length with age; or electrical instruments, which change readings with loss of magnetism.

Technical Errors are often caused by the use of instruments under conditions for which they are not standardized; for example, using a tape under too little or too much tension or in the hot sun.

Absolute Error is the total error in a measurement. It is expressed in the same units as the measurement; *i. e.*, 150 ft.-maximum error 3 ft.; 3 ft. is the absolute error.

Relative Error is the part of the measurement constituted by the error. It is a ratio of error to measurement, and is a pure number and has no units; *i. e.*, $\text{R. E.} = \frac{3\text{ ft.}}{150\text{ ft.}} = \frac{3}{150} = 0.02.$

Percentage Error is the percentage of the measurement constituted by the error. It is the relative error multiplied by 100; *i. e.*, $\text{P. E.} = \frac{3}{150} \times 100 = 2\%.$

If we are to know the reliability of a calculated result, we must take account of the uncertainty of each factor involved in the calcu-

lations. The uncertainty of a measurement is estimated after a consideration of the observer's errors, instrumental errors, and technical errors which may be involved. The uncertainty of a measurement is usually recorded after the measurement in the form of the estimated absolute error; *i.e.*, 154.1 ± 0.2 ft.

Uncertainties due to technical and observer's errors may be very high compared to the interval of graduation of a good instrument, so that some of the digits in the last decimal places may be of no significance. Consider the example of a surveyor's steel tape, 100 ft. long with an interval of graduation of 0.01 ft. Here it will be possible to estimate the reading to 0.001 ft. A reading of 95.382 ft. may be recorded but, unless the conditions of taking the measurement were ideal (*i.e.*, the tape kept absolutely in a straight line and at calibration temperature and tension), the uncertainty may be as high as 0.1 ft. In this case, the measurement might be stated as 95.382 ± 0.1 ft., except that, if the digit, 3, is in doubt by 1, the digit, 8, is in doubt by 10, and the digit, 2, is in doubt by 100. There is no point in keeping the last two digits, since they are so much in doubt. The digit, 8, does have some significance, and since 0.38 is nearer to 0.4 than to 0.3, the 3 is increased to 4 as the digit, 8, is discarded.

The following rules will be found helpful in stating the reliability of a result:

a In discarding unreliable digits, increase the last figure retained by one, if the following figure (one of those being discarded) is 5 or over.

b When finding an average or giving a numerical result of a calculation, keep in the answer only those digits which are certain, or in doubt by 9 or less.

(The figures remaining in an answer according to rules **a** and **b** are known as *significant figures*.)

c When the result is obtained by addition or subtraction, the *absolute* error in the result is the algebraic sum of the *absolute* errors in the quantities added or subtracted.

d When the result is obtained by multiplication or division, the *percentage* error in the result is the algebraic sum of the *percentage* errors in the quantities multiplied or divided.

When only a few measurements are involved in a calculation, generally the most pessimistic combination of errors is considered; *i.e.*, all the errors are considered to be acting in such a way as to make the resultant error a maximum. When many measurements are involved, however, some of the errors will be compensating, so that the error might justifiably be recorded as less than the most pessimistic case. Statistical methods must be applied to obtain an exhaustive prediction of the error one might expect from the results of a great many measurements.

Solutions to Exercises and Problems in Issue 8

ELEMENTS OF THE CALCULUS

VARIABLES AND LIMITS

VARIABLES

1 $f(x+h)=x^3+3x^2h+3xh^2+h^3$

2 $f(x+10)-f(x)=\log(x+10)-\log x$
$=\log\frac{x+10}{x}$

3 $f(x,x)=x^2x-x^2x=0$

4 $f\left(x+\frac{\pi}{6}\right)+f\left(x-\frac{\pi}{6}\right)=\sin\left(2x+\frac{\pi}{3}\right)+$
$\sin\left(2x-\frac{\pi}{3}\right)=\sin 2x\cos\frac{\pi}{3}+\cos 2x\sin\frac{\pi}{3}+$
$\sin 2x\cos\frac{\pi}{3}-\cos 2x\sin\frac{\pi}{3}=\sin 2x$ (since $\cos 60°=0.5$)

5 $f\left(x+\frac{\pi}{6}\right)+f\left(x-\frac{\pi}{6}\right)=\cos\left(2x+\frac{\pi}{3}\right)+$
$\cos\left(2x-\frac{\pi}{3}\right)=\cos 2x\cos\frac{\pi}{3}-\sin 2x\sin\frac{\pi}{3}+$
$\cos 2x\cos\frac{\pi}{3}+\sin 2x\sin\frac{\pi}{3}=\cos 2x$

LIMITS

6 $f(x)=\frac{1-x^2}{1-x}=\frac{(1-x)(1+x)}{(1-x)}=1+x=2$ at $x=1$

7 $\lim\frac{\log(1+x)}{e^x-1}=\lim\frac{x}{x}=1$ as $x\to 0$

8 $\lim\frac{1-\cos x}{2-e^x-e^{-x}}=$
$\lim\frac{1-(1-x^2\ldots)^{\frac{1}{2}}}{2-(1+x+\frac{1}{2}x^2\cdots)-(1-x+\frac{1}{2}x^2\cdots)}=-\frac{1}{2}$

DERIVATIVES

SUMS

9 $3x^2+4x$

10 $6x-1$

11 $3x^2+6x-1$

12 $\cos x-\sin x$

PRODUCTS

13 4

14 $9x^2+4x-1$

15 x

16 $4x+8$

17 $-4x^3-3x^2+12x$

18 $-2\sin x$

19 $\frac{dy}{dx}=(1-x)e^x-e^x=-xe^x$

20 $\frac{dy}{dx}=2a-2ax=2(a-x)$

21 $\frac{dy}{dx}=2(a-x)$

22 $16x^3+6x^2+6x$

QUOTIENTS

23 $\frac{dy}{dx}=\frac{(2x-1)-(x+2)2}{(2x-1)^2}=\frac{-5}{(2x-1)^2}$

24 $\frac{1}{(x+1)^2}$

25 $\frac{13}{2(3x+1)^2}$

26 $\frac{dy}{dx}=\frac{(1+x)-(x-1)}{(1+x)^2}=\frac{2}{(1+x)^2}$

27 $\frac{dy}{dx}=\frac{\cos^2x+\sin^2x}{\cos^2x}=\frac{1}{\cos^2x}=\sec^2 x$

POWERS

28 $(6x+9)(x^2+3x+4)^2$

29 $24(x-1)(2x^2-4x+2)^5$

30 $8x(x^2-1)^3$

31 $-2(1+x)^{-3}$

32 $16(x-1)(x^2-2x-2)$

33 $\frac{dy}{dx}=2\sin x\cos x=\sin 2x$

DIFFERENTIAL NOTATION

34 $\frac{dy}{dx}=\frac{\sin x}{\cos^2 x}$; $dy=\sec x\tan x\,dx$

35 $2\tan x\,d(\tan x)=2\sec x\,d(\sec x)=$
$2\sec^2x\tan x\,dx$ by 34. Hence, $d(\tan x)=\sec^2x\,dx$

36 $\frac{dx}{1+x^2}$

37 $e^x dx$

38 $f(x+\Delta x)$

39 $4ax^3$

40 $3x_1^2$; slope $=12$

41 (a) $\frac{1-x^2}{(x^2+1)^2}$
(b) $15x^2-4x+3$
(c) $6x^2+2x+2$

42 (a) $y=x$ (b) $y=\sqrt{x}$

HIGHER DERIVATIVES

43 $y=x^6$; $\frac{dy}{dx}=6x^5$; $\frac{d^2y}{dx^2}=30x^4$

44 2

45 $12ax^2+6bx+2c$

46 24

47 $-\frac{24}{x^5}$

48 $6x$; $24x$

MAXIMA AND MINIMA

49 Minimum $s=2\sqrt{c}$ at $x=\sqrt{c}$, $y=\sqrt{c}$

50 Area $=\frac{\pi D^2}{2}+\pi Dh=A$; Volume $=\frac{\pi D^2h}{4}$; hence $h=\frac{4V}{\pi D^2}$ and $A=\frac{\pi D^2}{2}+\frac{4V}{D}$; $\frac{dA}{dD}=\pi D-\frac{4V}{D^2}=0$ at min. Thus at min., $\pi D^3=4V$; $D=\sqrt[3]{\frac{4V}{\pi}}=\sqrt[3]{\frac{20000}{\pi}}$. $h=\frac{4V}{\pi}\left(\frac{4V}{\pi}\right)^{-\frac{2}{3}}=\left(\frac{4V}{\pi}\right)^{\frac{1}{3}}$. Area will be a minimum when $D=h=18.5$ ft.

51 $r=$radius, $h=$height, $V=$volume, $A=$area, $B=$area of hexagon circumscribed about r. Then $V=\pi r^2h$ or $h=\frac{V}{\pi r^2}$. $A=2B+2\pi rh=$
$12r^2\tan 30°+\frac{2V}{r}$. $\frac{dA}{dr}=24r\tan 30°-\frac{2V}{r^2}=0$ at min. Hence at min., $24r\tan 30°=\frac{2V}{r^2}$;
$r^3=\frac{2V}{24\tan 30°}$; $r=\sqrt[3]{\frac{V}{12\tan 30°}}$; $h=\frac{V}{\pi r^2}=$
$\frac{1}{\pi}\sqrt[3]{12^2V\tan^2 30°}$; $\frac{r}{h}=\frac{\pi}{12\tan 30°}=\frac{\pi\sqrt{3}}{12}$

52 $x^2+y^2=r^2=16$; $P=4x+4y$; $y=\sqrt{16-x^2}$;
$P=4x+4\sqrt{16-x^2}$; $\frac{dP}{dx}=4-\frac{4x}{\sqrt{16-x^2}}=0$.
$x=2\sqrt{2}$, $y=2\sqrt{2}$, or oblong is a square $4\sqrt{2}$ by $4\sqrt{2}$.

53 Let $a=$radius of paper blank, $x=$radius base of cone. $V=4\pi x^2\frac{\sqrt{a^2-x^2}}{3}$;
$\frac{dv}{dx}=\frac{8\pi x\sqrt{a^2-x^2}}{3}-\frac{4\pi x^3}{3\sqrt{a^2-x^2}}=0$; at max.,
$\frac{8\pi x(a^2-x^2)}{3}=\frac{4\pi x^3}{3}$. $x=\frac{1}{3}a\sqrt{6}=$radius of base;
$\sqrt{a^2-x^2}=\frac{1}{3}a\sqrt{3}=$height of cone.

54 $y=\sin x\cos x=\frac{\sin 2x}{2}$; $\frac{dy}{dx}=\cos 2x=0$; $2x=\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc.; $x=\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, . . . or $\frac{(2n-1)\pi}{4}$ where $n=1, 2, 3$, etc.

55 $t=$time in hours; $D=$distance apart (nautical miles). $D^2=(10t)^2-(20-12t)^2-$
$2\cdot 10t(20-12t)\cos 30°$ (Law of Cosine).
$2D\frac{dD}{dt}=t(200+288+480\cos 30°)-$
$(480-400\cos 30°)=0$ at min. Hence at min., $t=$
$\frac{120+100\cos 30°}{122+120\cos 30°}$ hrs. $=55$ min.

PARTIAL DIFFERENTIATION

56 h = altitude; 2θ = vertical angle; r = radius of base $= h \tan\theta$; s = slant height $= h\sec\theta$; A = surface $= \pi r s = \pi h^2 \frac{\sin\theta}{\cos^2\theta}$; V = volume $= \frac{1}{3}\pi r^2 h = \frac{\pi}{3}h^3\tan^2\theta$; $ds = \sec\theta\, dh + h\frac{\sin\theta}{\cos^2\theta}d\theta$;

$dA = 2\pi h\frac{\sin\theta}{\cos^2\theta}dh + \pi h^2 \frac{1+2\tan^2\theta}{\cos\theta}d\theta$;

$dV = \pi h^2\tan^2\theta\, dh + \frac{2\pi}{3}h^3\frac{\tan\theta}{\cos^2\theta}d\theta$. Substituting $h=5$, $dh=0.2$; $\theta=30°$, $d\theta=\frac{1}{2}°=0.00873$ radians;

$\sin\theta=\frac{1}{2}$, $\cos^2\theta=\frac{3}{4}$, $\tan^2\theta=\frac{1}{3}$, we get $V=\frac{\pi}{3}\cdot 5^3\cdot\frac{1}{3}=$ 43.6 cu. ft.

$dV=\pi\cdot 5^2\cdot\frac{1}{3}\cdot 0.2+\frac{2\pi}{3}\cdot 5^3\cdot\frac{4}{3\sqrt{3}}\cdot 0.00873=6.8$ cu. ft.

Similarly, $A=52.4$ sq. ft., $s=5.77$ ft., $dA=5.5$ sq. ft., $ds=0.26$ ft. Volume $=V+dV=50.4$ cu. ft.; Area $=A+dA=57.9$ sq. ft.; Slant Height $=s+ds=6.03$ ft.

57 $$\frac{\partial u}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)+\frac{2x}{2\sqrt{x^2+y^2}}=-\frac{y}{x^2+y^2}+\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=-\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}-\frac{1}{2}\frac{x\cdot 2y}{\sqrt{x^2+y^2}^3}=\frac{y^2-x^2}{(x^2+y^2)^2}-\frac{xy}{\sqrt{x^2+y^2}^3}$$

$$\frac{\partial u}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}+\frac{2y}{2\sqrt{x^2+y^2}}=\frac{x}{x^2+y^2}+\frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}-\frac{1}{2}\frac{y\cdot 2x}{\sqrt{x^2+y^2}^3}=\frac{y^2-x^2}{(x^2+y^2)^2}-\frac{xy}{\sqrt{x^2+y^2}^3}$$

and $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

INTEGRATION

58 $\frac{x^5}{5}+C$

59 $x^4-\frac{2}{3}x^3+\frac{5}{2}x^2+6x+C$

60 $\frac{x^2}{2}+C$

61 $\frac{2}{3}x^{\frac{3}{2}}+C$

62 $\log_e x+C$

63 $\frac{1}{3}(a+bx)^3+C$

64 $uv+C$

65 x^2+x+C, a parabola

MEASURING ROD

1 $\frac{dy}{dx}=6x^2-8x+1$. Setting $\frac{dy}{dx}$ equal to 0 and solving, we find max./min. points at $x=\frac{2}{3}\pm\frac{\sqrt{10}}{6}$. $\frac{d^2y}{dx^2}=12x-8$, and the point of inflection occurs at $x=\frac{2}{3}$.

2 $\frac{dy}{dx}=\frac{1}{2}$. The angle is $\tan^{-1}\frac{1}{2}=26°34'$.

3 The stone travels in the arc of a parabola. Take the point from which it was thrown as the origin, and the downward direction as the positive y direction. $y=\frac{1}{2}gt^2=16t^2$; $x=vt=100t$; $y=\frac{16}{10000}x^2$; $\frac{dy}{dx}=\frac{32x}{10000}$ at the point (100,16); $\frac{dy}{dx}=\frac{32}{100}=0.32$; $\therefore$ the stone strikes the water at $\tan^{-1}0.32=17°45'$.

4 Take the lowest point of the cable as the origin. Then its equation is $y=\frac{x^2}{250}$ and $\frac{dy}{dx}=\frac{x}{125}$. At the point (100,40), $\frac{dy}{dx}=\frac{100}{125}=0.80$. The angle with the horizontal is $\tan^{-1}0.80$, but the pillar is vertical; therefore, the angle required is $(90°-\tan^{-1}0.80)=51°20'$.

5 $\frac{dy}{dx}=1-2x$. We have either a max. or min. when $x=\frac{1}{2}$. Inspection will show that it is a max. and $y=\frac{1}{4}$.

6 Place the rectangle so that one of its sides coincides with a side of the triangle. Call the length of this side of the rectangle y and the length of the other side x. Consider one of the small triangles exterior to the rectangle. $x=\sqrt{3}\left(\frac{10-y}{2}\right)$. The area of the rectangle is given by $A=xy$. Substituting, we get: $A=5\sqrt{3}y-\frac{\sqrt{3}y^2}{2}$; $\frac{dA}{dy}=5\sqrt{3}-\sqrt{3}y$. Setting $\frac{dA}{dy}$ equal to 0, we find that we have a max. area when $y=5$, $x=2.5\sqrt{3}$.

7 The centers of the ellipse and inscribed rectangle will obviously be the same. Call the dimensions of the rectangle $2x$ and $2y$ and consider that portion of the figure in the first quadrant, $A=xy$. $\frac{x^2}{441}+\frac{y^2}{225}=1$; $y=15\sqrt{1-\frac{x^2}{441}}$; $y=\frac{5}{7}\sqrt{441-x^2}$.
Substituting:
$A=\frac{5}{7}x\sqrt{441-x^2}$; $\frac{dA}{dx}=\frac{5}{7}\sqrt{441-x^2}-\frac{5x^2}{7\sqrt{441-x^2}}$.
Setting $\frac{dA}{dx}$ equal to 0, $x=\frac{21}{2}\sqrt{2}$. Substituting in the equation for the ellipse: $\frac{y^2}{225}=\frac{1}{2}$; $y=\frac{15}{2}\sqrt{2}$.
$A=\frac{15\cdot 21}{2}$ and the total area $=4A=630$ sq. in.

8 The formula for the volume of the cylinder is $V=\pi r^2h$. Draw a cross-section diagram. By similar triangles:
$$\frac{r}{2}=\frac{6-h}{6};\ h=6-3r;\ V=\pi(6r^2-3r^3);$$
$\frac{dV}{dr}=3\pi(4r-3r^2)=0$. $r=1\frac{1}{3}$ in., $h=2$ in.

9 $S=2\pi(r^2+rh)$; $V=116=\pi r^2h$; $h=\frac{116}{\pi r^2}$
$S=2\pi\left(r^2+\frac{116}{\pi r}\right)$; $\frac{dS}{dr}=2\pi\left(2r-\frac{116}{\pi r^2}\right)$; when $\frac{dS}{dr}=0$, $r=\sqrt[3]{\frac{58}{\pi}}=2.64$ in.; $h=2\sqrt[3]{\frac{58}{\pi}}=5.29$ in.

10 Let height be y and side of base x.
$V=x^2y$; $S=84=x^2+4xy$
$y=\frac{84-x^2}{4x}$; $V=21x-\frac{x^3}{4}$
$\frac{dV}{dx}=21-\frac{3x^2}{4}=0$;
$x=2\sqrt{7}$, $y=\sqrt{7}$.

11 V = volume $=\frac{\pi r^2h}{3}$; $h=\frac{3V}{\pi r^2}$ (h, r = alt., radius)
A = area $=\pi r\sqrt{h^2+r^2}$; $\frac{dh}{dr}=-\frac{6V}{\pi r^3}=-\frac{2h}{r}$
$\frac{dA}{dr}=\pi\sqrt{h^2+r^2}+\frac{\pi r}{\sqrt{h^2+r^2}}\left(r+h\frac{dh}{dr}\right)=0$ at min. area. Hence at min. area,
$h^2+r^2+r\left(r-\frac{2h^2}{r}\right)=0$, or $h^2=2r^2$. Hence,
$$\frac{9V^2}{\pi^2r^4}=2r^2 \text{ and } r^2=\left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{3}},\ h^2=2\left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{3}}$$
$$\text{Area}=\pi r^2\left(1+\frac{h^2}{r^2}\right)^{\frac{1}{2}}=\pi\left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{3}}3^{\frac{1}{2}}=\sqrt{3}\sqrt[3]{2\pi\cdot 69^2}$$
$=53.8$ sq.ft.

12 Let x be length of side of squares.
Volume of box $=V=x(40-2x)^2$.
$\frac{dV}{dx}=12x^2-320x+1600$; when $\frac{dV}{dx}=0$, $x=20$

MEASURING ROD
(continued)

or $\frac{20}{3}$; only the latter is possible.

$V=\frac{20}{3}\left(\frac{80}{3}\right)^2=4740\frac{20}{27}$ cu. in.

13 Reducing the units to inches, the equation of the ellipse becomes $\frac{x^2}{36}+\frac{y^2}{25}=1$.

Let the depth be $2x$ and the breadth $2y$. $S=K.4x^2.2y=Cx^2y$ (S = strength, K,C are constants). But $x^2=36-\frac{36y^2}{25}$; hence,

$S=C\left(36y-\frac{36y^3}{25}\right)$. $\frac{dS}{dy}=C\left(36-\frac{108y^2}{25}\right)$;

when $\frac{dS}{dy}=0$, $y=\frac{5\sqrt{3}}{3}$, $x=2\sqrt{6}$. The dimensions are, breadth 5.77 in., depth 9.80 in.

14 Let x be speed and y rate of using coal. Then $y=\frac{2x^2}{125}+1$. A, the amount of coal used, equals yt, t being the time; $t=\frac{500}{x}$;

$A=\frac{500y}{x}=\frac{1000x}{125}+\frac{500}{x}$; $\frac{dA}{dx}=8-\frac{500}{x^2}$;

when $\frac{dA}{dx}=0$, $x=\frac{5}{2}\sqrt{10}=7.9$ mi. per hr.

15 $y=x-\frac{32x^2}{1800^2}$; $\frac{dy}{dx}=1-\frac{64x}{1800^2}$; when $\frac{dy}{dx}=0$,

$x=\frac{1800^2}{64}$. Substituting in the first equation,

$y=\frac{1800^2}{64}-\frac{1800^2}{2\cdot 64}=25312.5$ ft.

16 Let distance from C be x and from D, $12-x$. If L is length of main,

$L=\sqrt{9^2+(12-x)^2}+\sqrt{6^2+x^2}$

$=\sqrt{225-24x+x^2}+\sqrt{36+x^2}$

$\frac{dL}{dx}=\frac{x-12}{\sqrt{225-24x+x^2}}+\frac{x}{\sqrt{36+x^2}}$

When $\frac{dL}{dx}=0$, $\frac{x^2-24x+144}{x^2}=\frac{225-24x+x^2}{36+x^2}$

$5x^2+96x-576=0$; $x=4.8$ mi.

17 x and y the dimensions, L = length of fence, A = area.

$L=2x+2y$; $A=xy=324$; $y+x\frac{dy}{dx}=0$; $\frac{dy}{dx}=-\frac{y}{x}$

$\frac{dL}{dx}=2+\frac{dy}{dx}=2-\frac{y}{x}=0$, or $x=y=\sqrt{324}=18$ yd.

18 Limit $\frac{\Delta V}{\Delta r}=\frac{dV}{dr}=4\pi r^2=A$

19 x = edge, S = surface, V = volume, of cube. Then

$S=x^2$ $\qquad V=x^3$

$dS=2x\,dx$ $\qquad dV=3x^2dx$

$\frac{dS}{S}=\frac{2dx}{x}$ $\qquad \frac{dV}{V}=\frac{3dx}{x}$

$\frac{dx}{x}=0.25\%$; hence $\frac{dS}{S}=0.50\%$, $\frac{dV}{V}=0.75\%$.

20 $V=\pi r^2h$ ($r=2''$, $h=7''$)

$dV=2\pi rhdr+\pi r^2dh$ ($dr=dh=0.01''$)

$=\frac{28\pi}{100}+\frac{4\pi}{100}=1.01$ cu. in. of possible error.

To express the same result as a percentage, we may write $\frac{dV}{V}=\frac{2\pi rhdr+\pi r^2dh}{\pi r^2h}$

$=\frac{2dr}{r}+\frac{dh}{h}=\frac{2}{200}+\frac{1}{700}=1\frac{1}{7}\%$

21 $dV=\frac{\pi d^2.dd}{2}$; $\frac{dV}{V}=\frac{3dd}{d}=2\%$; hence $\frac{dd}{d}=0.67\%$

22 $S=\frac{1}{2}gt^2$; $\frac{dS}{dt}=V=gt$

23 $\frac{dS}{dt}=-\frac{1}{\sqrt{t}}+4t^3-18t^2=v$;

when $t=8$, $v=896-\frac{\sqrt{2}}{4}=895.65$, the velocity.

$\frac{dv}{dt}=\frac{d^2S}{dt^2}=a=\frac{1}{2t\sqrt{t}}+12t^2-36t$;

when $t=8$, $a=480+\frac{\sqrt{2}}{64}=480.02$, the accel.

24 $A=\pi r^2$; $\frac{dA}{dt}+2\pi r\frac{dr}{dt}$; $\frac{dr}{dt}=0.01$

$\frac{dA}{dt}=0.08\pi=0.25$ sq. in. per min.

25 Let t be the time in hours and S the distance between plane and battery.

$S=\sqrt{\frac{1}{4}+1+(200t)^2}$; $\frac{dS}{dt}=\frac{40000t}{\sqrt{1.25+40000t^2}}$

$t=\frac{1}{60}$; substituting, $\frac{dS}{dt}=189.6+$ m.p.h.

26 $V=\frac{1}{3}\pi r^2x$ and $r=\frac{x}{2}$, or $V=\frac{\pi x^3}{12}$; $\frac{dV}{dt}=\frac{\pi x^2dx}{4dt}$;

$\frac{dx}{dt}=\frac{1}{6}$, so $\frac{dV}{dt}=\frac{25\pi}{24}$ cu. ft. per min.

27 $A=\int_a^b ydx=\int_0^1(2-2x)dx=[2x-x^2]_0^1=1$

28 $y=0$ when $x=\pm\sqrt{2}$

$A=2\int_0^{\sqrt{2}}(2-x^2)dx=2\left[2x-\frac{x^3}{3}\right]_0^{\sqrt{2}}=\frac{8\sqrt{2}}{3}$

29 $A=\int_0^2\sqrt{4x-x^2}dx-\int_0^2\frac{x^2}{2}dx$

$=\left[\frac{x-2}{2}\sqrt{4x-x^2}+2\sin^{-1}\frac{x-2}{2}-\frac{x^3}{6}\right]_0^2$

$=2\cdot 2\pi-\frac{4}{3}-2\cdot\frac{3\pi}{2}=\pi-\frac{4}{3}$

30 Let the function be $y=f(x)$. Then the difference will be dy; $\frac{dy}{y}=k$. Since $\int\frac{dy}{y}=\log Cy$, we know that the table was a logarithm table.

31 $x_1=\frac{1}{6}\int_0^3 xydx=\frac{1}{6}\int_0^3\left(4x-\frac{4x^2}{3}\right)dx=1$

$y_1=\frac{1}{6}\int_0^4 xydy=\frac{1}{6}\int_0^4(3y-\frac{3}{4}y^2)dx=\frac{4}{3}$

Center of gravity is at pt. $(1,1\frac{1}{3})$

32 The length of the semicircle is $\pi r=4\pi=12.57$; of the arc of the parabola $y=x^2$,

$2\int_{x=0}^{x=4}ds=2\int_0^4\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$. Since $\frac{dy}{dx}=2x$,

$P=2\int_0^4\sqrt{1+4x^2}dx$

$=4\left[\frac{x\sqrt{x^2+\frac{1}{4}}}{2}+\frac{1}{8}\log_e(x+\sqrt{x^2+\frac{1}{4}})\right]_0^4$

$=4\,[\sqrt{65}+\frac{1}{8}\log_e(8+\sqrt{65})]=33.63$ in.

Perimeter $=33.63+12.57=46.2$ in.

33 $A=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}dx=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a=\frac{4b}{a}\left[\frac{a^2\pi}{4}-0\right]=\pi ab$

34 Set up the circle with center at (0,3): $x^2+y^2-6y=0$. The pressure (dF) on a small horizontal strip is equal to the area (dA) of the strip times the depth (h) of the strip times density (d) of the liquid. In the first quadrant: $d=55$, $dA=xdy$, $h=3-y$

$F=110\int_0^3(\sqrt{6y-y^2})(3-y)dy$

$=55\int_0^3\sqrt{6y-y^2}d(6y-y^2)=990$ lb.

35 The balloon can be considered the ellipse, $\frac{x^2}{100}+\frac{y^2}{16}=1$, revolved about the OX-axis. The ellipsoid is composed of cylindrical strips of volume πy^2dx. $V=2\pi\int_0^{10}y^2dx$

$=32\pi\int_0^{10}\left(1-\frac{x^2}{100}\right)dx=32\pi\left[x-\frac{x^3}{300}\right]_0^{10}$

$V=\frac{640}{3}\pi$ cu. ft. (notice that $V=\frac{4}{3}\pi abc$)

Tables and Formulas

Note: $\frac{2}{0}$, $\frac{3}{0}$, $\frac{4}{0}$, etc., indicate the number of ciphers that follow the decimal. Thus, one inch = 0.005051 rods = 0.00001578 statute miles.

TABLE XLIII
EQUIVALENT LENGTHS

	INCHES (*in.*)	FEET (*ft.*)	YARDS (*yd.*)	RODS (*r.*)	CHAINS (*ch.*)	STATUTE MILES	NAUTICAL MILES	METERS (*m.*)	KILOMETERS (*km.*)
1 in.	1	0.08333	0.02778	$0.\frac{2}{0}5051$	$0.\frac{2}{0}1263$	$0.\frac{4}{0}1578$	$0.\frac{4}{0}1371$	0.02540	$0.\frac{4}{0}2540$
1 ft.	12	1	0.33333	0.06061	0.01515	$0.\frac{3}{0}1894$	$0.\frac{3}{0}1645$	0.30480	$0.\frac{3}{0}3048$
1 yd.	36	3	1	0.18182	0.04545	$0.\frac{3}{0}5682$	$0.\frac{3}{0}4934$	0.91440	$0.\frac{3}{0}9144$
1 r.	198	16.5	5.5	1	0.25	$0.\frac{2}{0}3125$	$0.\frac{2}{0}2714$	5.02921	$0.\frac{2}{0}5029$
1 ch.	792	66	22	4	1	0.01250	0.01085	20.1168	0.02012
1 S.M.	63360	5280	1760	320	80	1	0.86839	1609.35	1.60935
1 N.M.	72962.5	6080.20	2026.73	368.497	92.1243	1.15155	1	1853.25	1.85325
1 m.	39.37	3.28083	1.09361	0.19884	0.04971	$0.\frac{3}{0}6214$	$0.\frac{3}{0}5396$	1	0.001
1 km.	39370	3280.83	1093.61	198.838	49.7096	0.62137	0.53959	1000	1

TABLE XLIV
EQUIVALENT AREAS

	SQ. IN.	SQ. FT.	SQ. YD.	SQ. RODS	ACRES	SQ. MI.	SQ. CM.	SQ. M.	SQ. KM.
1 sq. in.	1	$0.\frac{2}{0}6944$	$0.\frac{3}{0}7716$	$0.\frac{4}{0}2551$	$0.\frac{6}{0}1594$	$0.\frac{9}{0}2491$	6.452	$0.\frac{3}{0}6452$	$0.\frac{9}{0}6452$
1 sq. ft.	144	1	0.11111	$0.\frac{2}{0}3673$	$0.\frac{4}{0}2296$	$0.\frac{7}{0}3587$	929.0	0.09290	$0.\frac{7}{0}9290$
1 sq. yd.	1296	9	1	0.03306	$0.\frac{3}{0}2066$	$0.\frac{6}{0}3228$	8361.3	0.83613	$0.\frac{6}{0}8361$
1 sq. r.	39204	272.25	30.25	1	0.00625	$0.\frac{5}{0}9766$	252930	25.2930	$0.\frac{4}{0}2529$
1 acre	6272640	43560	4840	160	1	$0.\frac{2}{0}1563$	40468700	4046.87	$0.\frac{2}{0}4047$
1 sq. mi.	....	27878400	3097600	102400	640	1	....	2589999	2.59000
1 sq. cm.	0.1550	0.001076	$0.\frac{3}{0}1196$	$0.\frac{5}{0}3954$	....	....	1	$0.\frac{3}{0}1$	$0.\frac{9}{0}1$
1 sq. m.	1550.00	10.7639	1.19599	0.03954	$0.\frac{3}{0}2471$	$0.\frac{6}{0}3861$	10000	1	$0.\frac{5}{0}1$
1 sq. km.	....	10763867	1195985	39536.6	247.104	0.38610	....	1000000	1

TABLE XLV
EQUIVALENT VELOCITIES AND ACCELERATIONS

	FT./SEC.	MI./HR.	KNOTS	METERS/SEC.	KM./HR.
1 ft./sec.	1	0.68182	0.59209	0.30480	1.09728
1 mi./hr.	1.46667	1	0.86839	0.44704	1.60935
1 knot	1.68894	1.15155	1	0.51479	1.85325
1 m./sec.	3.28083	2.23693	1.94254	1	3.6
1 km./hr.	0.91134	0.62137	0.53959	0.27778	1

TABLE XLVI
EQUIVALENT VOLUMES

	CUBIC INCHES	CUBIC FEET	CUBIC YARDS	LIQUID QUARTS	LIQUID GALLONS	BUSHELS	CUBIC DECI-METERS (*liters*)
1 cu. in.	1	$0.\frac{3}{0}5787$	$0.\frac{4}{0}2143$	0.01732	$0.\frac{2}{0}4329$	$0.\frac{3}{0}4650$	0.01639
1 cu. ft.	1728	1	0.03704	29.9221	7.48055	0.80356	28.3170
1 cu. yd.	46656	27	1	807.896	201.974	21.6962	764.559
1 liq. qt.	57.75	0.03342	$0.\frac{2}{0}1238$	1	0.25	0.02686	0.94636
1 liq. gal.	231	0.13368	$0.\frac{2}{0}4951$	4	1	0.10742	3.78543
1 bu.	2150.42	1.24446	0.04609	37.2368	9.3092	1	35.2393
1 cu. dm.	61.0234	0.03531	$0.\frac{2}{0}1308$	1.05668	0.26417	0.02838	1

Liquid quarts and gallons $\times$ 0.86 = dry quarts and gallons.

TABLE XLVII

EQUIVALENT FORCES, WEIGHTS, AND MASSES

(45° *Lat. at sea level*)

	Grains	Ounces (*avoir.*)	Pounds (*avoir.*)	Short tons (*2000 lb.*)	Dynes	Newtons	Grams	Kg.
1 grain	1	$0.\overset{2}{0}2286$	$0.\overset{3}{0}1429$	$0.\overset{7}{0}7143$	63.546	$0.\overset{3}{0}6355$	0.0648	$0.\overset{4}{0}6480$
1 ounce	437.5	1	0.06250	$0.\overset{4}{0}3125$	28013.8	0.28014	28.35	0.02835
1 pound	7000	16	1	0.00050	444822	4.44822	453.59	0.45359
1 ton	14000000	32000	2000	1	....	8896.44	907185	907.185
1 dyne	0.015737	$0.\overset{4}{0}3534$	$0.\overset{5}{0}22481$	....	1	$0.\overset{4}{0}1000$	$0.\overset{2}{0}10197$	$0.\overset{5}{0}10197$
1 newton	1573.7	3.53397	0.22481	$0.\overset{3}{0}11241$	100000	1	101.97	0.10197
1 gram	15.4324	0.035274	$0.\overset{2}{0}22046$	$0.\overset{5}{0}1102$	980.67	$0.\overset{2}{0}98067$	1	0.001000
1 kg.	15432.4	35.2740	2.20462	$0.\overset{2}{0}1102$	980665	9.80665	1000	1

TABLE XLVIII

EQUIVALENT PRESSURES AND STRESSES

(45° *Lat. at sea level*)

	Lbs./in.^{2}	Lbs./ft.^{2}	Atmospheres	In. of Mercury	Mm. of Mercury	Dynes/cm.^{2}	Kg./cm.^{2}
1 lb./in.^{2}	1	144	0.06804	2.03588	51.7116	68944	0.07031
1 lb./ft.^{2}	$0.\overset{2}{0}6944$	1	$0.\overset{3}{0}4725$	0.01414	0.35911	478.80	$0.\overset{3}{0}4882$
1 atmos.	14.6969	2116.35	1	29.9212	760	1013200	1.03329
Mercury							
1 in.	0.49119	70.7310	0.03342	1	25.4001	33863.9	0.03453
1 mm.	0.01934	2.78468	$0.\overset{2}{0}1316$	0.03937	1	1332	$0.\overset{2}{0}1360$
1 dyne/cm.^{2}	$0.\overset{4}{0}14504$	$0.\overset{2}{0}20886$	$0.\overset{6}{0}98692$	$0.\overset{4}{0}29530$	$0.\overset{3}{0}75006$	1	$0.\overset{5}{0}101971$
1 kg./cm.^{2}	14.2234	2048.17	0.96778	28.9572	735.514	478.78	1

TABLE XLIX

DENSITY AND SPECIFIC GRAVITY

	Lbs./in.^{3}	Lbs./ft.^{3}	Grams/cm.^{3}	Specific gravity
1 lb./in.^{3}	1	1728	27.6797	27.6797
1 lb./ft.^{3}	$0.\overset{3}{0}5787$	1	0.01602	0.01602
1 gram/cm.^{3}	0.03613	62.4283	1	1
Sp. G.	0.03613	62.4283	1	1

TABLE L

EQUIVALENTS OF ENERGY, WORK, HEAT

	Ft.-lb.	Kg.-m.	Joules (*Absolute*)	H.P.-hr.	Kw.-hr.	B.T.U.	Calories (*Kg.-cal.*)
1 ft.-lb.	1	0.13826	1.35573	$0.\overset{6}{0}5051$	$0.\overset{6}{0}3766$	$0.\overset{2}{0}1285$	$0.\overset{3}{0}3239$
1 kg.-m.	7.23300	1	9.80597	$0.\overset{5}{0}3653$	$0.\overset{5}{0}2724$	$0.\overset{2}{0}9296$	$0.\overset{2}{0}2342$
1 joule	0.73761	0.10198	1	$0.\overset{6}{0}3725$	$0.\overset{6}{0}2778$	$0.\overset{3}{0}9480$	$0.\overset{3}{0}2389$
1 H.P.-hr.	1980000	273745	2684340	1	0.74565	2544.65	641.240
1 kw.-hr.	2655403	367123	3600000	1.34111	1	3412.66	859.975
1 B.T.U.	778.104	107.577	1054.90	$0.\overset{3}{0}3930$	$0.\overset{3}{0}2930$	1	0.25200
1 cal.	3087.77	426.900	4186.17	$0.\overset{2}{0}1559$	$0.\overset{2}{0}1163$	3.96832	1

1 International Joule = 1.00032 Absolute Joules

1 Erg = 10^{-7} × International Joule

TABLE LI
EQUIVALENTS OF POWER

	Ft.-lb./sec.	Watts (*Joules/sec.*)	H.P.	Kilowatts	B.T.U./sec.	Cal./sec. (*Kg.-cal./sec.*)
1 ft.-lb./sec.	1	1.35573	$0.{}_{0}^{2}1818$	$0.{}_{0}^{2}1356$	$0.{}_{0}^{2}1285$	$0.{}_{0}^{3}3237$
1 watt	0.73761	1	$0.{}_{0}^{2}1341$	0.001	$0.{}_{0}^{3}9480$	$0.{}_{0}^{3}2389$
1 H.P.	550	745.650	1	0.74565	0.70685	0.17812
1 kw.	737.612	1000	1.34111	1	0.94796	0.23888
1 B.T.U./sec.	778.104	1054.90	1.41474	1.05490	1	0.25200
1 cal./sec.	3087.77	4186.17	5.61412	4.18617	3.96832	1

SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS

In the following forms, x represents the independent variable, y the dependent variable; X, P, Q, etc., stand for functions of x; Y stands for a function of y, $A, B, C, \ldots$, are arbitrary constants, $a, b, c, \ldots$, constant coefficients, and $y', y'', \ldots$, stand for $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, etc.

	Equation	Solution
First Order Equations		
Linear	$y'+Py=Q$	$y=e^{-\int Pdx}(\int Qe^{\int Pdx}dx+C)$
Clairaut's[1]	$xy'+f(y')=y$	$y=Cx+f(C)$
Separable[2]	$Yy'+X=0$	$\int Ydy=C-\int Xdx$
Second Order Equations		
Linear	$y''+ay'+by=0$	$y=e^{-\frac{1}{2}at}(A\cos\sqrt{k}t+B\sin\sqrt{k}t)$ if $k>0$
		$y=e^{-\frac{1}{2}at}(A+Bt)$ if $k=0$
		$y=e^{-\frac{1}{2}at}(Ae^{\sqrt{-k}t}+Be^{\sqrt{-k}t})$ if $k<0$
		where $k=b-\frac{a^2}{4}$.
	$y''=X$	$y=C'+Cx+\iint Xdxdx$
	$y''=Y$	$x=C'+\int\frac{dy}{\sqrt{C+2\int Ydy}}$
Higher Order Equations		
Linear	$y^{(n)}+a_1y^{(n-1)}+a_2y^{(n-2)}+\ldots+a_{n-1}y'+a_ny=0$	$y=A_1e^{r_1x}+A_2e^{r_2x}+\ldots+A_ne^{r_nx}$ where $r_1, r_2\ldots r_n$ are the n roots of the algebraic equation $z^n+a_1z^{n-1}+a_2z^{n-2}+\ldots+a_n=0$

[1] Clairaut's form has a singular solution.

[2] In the equation, $y'=H(y, x)$, where $H(y, x)$ is a polynomial in x, y in which all terms have the same degree, substitution of the new dependent variable, v, given by $y=vx$, $dy=vdx+xdv$ will reduce the differential equation to the separable form in the variables x, v.